Faculty of Technology — Department of E.B. $\mathcal{S}$. & $\mathcal{T}$.

Exam of module : **Maths 1**
Duration : **1h. 30**
Tuesday 16/01/2024

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**Exercise 1.** *(Applications 4pts)Prove that the application f is bijective and give its inverse application* $f^{-1}$.

$$f : [-1,1] \longrightarrow [-1,1]$$
$$x \longmapsto \frac{2x}{1+x^2}.$$

**correction**

**a.** is f bijective ?

Let $y \in [-1,1]$, we solve the equation $y = f(x)$.

We have

$$y = f(x) \Leftrightarrow y = \frac{2x}{1+x^2}$$
$$\Leftrightarrow yx^2 - 2x + y = 0.$$

We solve the second-order equation with the discriminant
$\Delta = b^2 - 4ac = 4(1-y^2) \geq 0.$ 0.5pt
The solutions are

$x_1 = \frac{2-\sqrt{4(1-y)^2}}{2y} = \frac{1-\sqrt{1-y^2}}{y}$

and $x_2 = \frac{2+\sqrt{4(1-y^2)}}{2y} = \frac{1+\sqrt{1-y^2}}{y}$

$x_2 \notin [-1,1]$ because $1+\sqrt{1-y^2} \geq 1$ and $y \in [-1,1]$
$x_1 \in [-1,1]$ indeed
$x_1 = \frac{1-\sqrt{1-y^2}}{y} = \frac{y}{1+\sqrt{1+y^2}}$.
Since $1+\sqrt{1+y^2} \geq 1$ and $y \in [-1,1]$, then $x_1 \in [-1,1]$ 1pt+1pt
Therefore

$$\forall y \in [-1,1], \exists! x = \frac{1-\sqrt{1-y^2}}{y} \in [-1,1],\ y = f(x)$$

this shows that $f$ is bijective. 0.5pt
**b.** The inverse $f^{-1}$ of $f$. 01pt

$$\begin{aligned} f^{-1} : [-1,1] &\longrightarrow [-1,1] \\ y &\longmapsto f^{-1}(y) = \tfrac{1-\sqrt{1-y^2}}{y}. \end{aligned}$$

**Exercise 2.** *(Complex numbers 4pts) Give the algebraic, trigonometric and exponential form of the solutions of the following equation :*

$$z^2 - (1+2i)z + (i-1) = 0.$$

**correction**

Solve in $\mathbf{C}$ the equation : $z^2 - (1+2i)z + i - 1 = 0$.
The discriminant is :

$$\begin{aligned} \Delta = b^2 - 4ac &= (1+2i)^2 - 4(i-1) \\ \Delta &= 1 - 4 + 4i - 4i + 4 \\ \Delta &= 1. \ 1pt \end{aligned}$$

Then

$z_1 = \frac{1+2i+1}{2} = 1+i$ and $z_2 = \frac{1+2i-1}{2} = i$
The algebraic form : 1pt
$z_1 = 1+i$ and $z_2 = i$
The trigonometric form : 1pt
$|z_1| = \sqrt{2}$, $Arg(z_1) = \frac{\pi}{4}$ and $z_1 = \sqrt{2}(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4}))$
and
$|z_2| = 1$, $Arg(z_1) = \frac{\pi}{2}$ and $z_2 = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})$
The exponential form : 1pt
$z_1 = \sqrt{2}e^{i\frac{\pi}{4}}$ and $z_2 = e^{i\frac{\pi}{2}}$

**Exercise 3.** *(Sequences 6pts). We consider the numerical sequences $(u_n)$ such that :*
$u_0 > 1$ *and* $\forall n \in \mathbb{N},\ u_{n+1} = 2 - \frac{1}{u_n}$.

1. *Prove that :* $\forall n \in \mathbb{N},\ u_n > 1$.
2. *Study the monotony of this sequence.*
3. *Deduce that the sequence $(u_n)_n$ is convergente and specify its limit.*

**correction**

**1.** We prove by induction that : $\forall n \in \mathbb{N},\ u_n > 1$. 2pts
We denote $(H_n) : (u_n > 1)$
**a.** For $n = 0$ We have $u_0 > 1$, then $(H_n)$ is satisfied.
**b.** We assume that $(H_n)$ is satisfied for $n > 0$ and we demonstrate that $H_{n+1}$ is also satisfied. We have :
$u_{n+1} = 2 - \frac{1}{u_n} > 2 - \frac{1}{1}$ because $u_n > 1$
then $u_{n+1} > 1$ this shows that $H_{n+1}$ is satisfied.

Using the recurrence theorem, we deduce that $(H_n)$ is true for all $n \in \mathbb{N}$, then :

$$\forall n \in \mathbb{N}, \quad u_n > 1$$

**2.**The monotony. 2pts

We study the sign of $(u_{n+1} - u_n)$, we have :

$u_{n+1} - u_n = 2 - \frac{1}{u_n} - u_n = \frac{2u_n - 1 - (u_n)^2}{u_n} = -\frac{(u_n - 1)^2}{u_n}$

Since $u_n > 1 > 0$, we deduce that

$$\forall n \in \mathbb{N}, \quad (u_{n+1} - u_n) < 0$$

This shows that $(u_n)_n$ is decreasing

**3.a.** Convergence of the seqeence $(u_n)_n$. 1 pt

Since this sequence is decreasing and bounded below (by 1) we deduce that it is convergente

**3.b.** The limit. 1 pt

Let l= $\lim\limits_{n \to +\infty} U_n$, and since

$$\forall n \in \mathbb{N}, \; u_{n+1} = 2 - \frac{1}{u_n}$$

By taking the limit, we obtain :

$$l = 2 - \frac{1}{l}$$

and the only solution to this equation is : $(l = 1)$, then : $\lim\limits_{n \to +\infty} U_n = 1$

**Exercise 4.** *(Real functions 6pts).*

***1.*** *Using the intermidiate value theorem, prove that the equation* $(x^3 + x^2 - 4x + 1 = 0)$ *has at least one solution* $x_0 \in ]0, 1[$

***2.*** *Using l'Hopital's Rule ,calculate the following limits :* $\lim\limits_{x \to 0} \dfrac{ln(1 + x^2)}{sin^2(x)}, \quad \lim\limits_{x \to 0} \dfrac{arctan(x)}{x + x^2}$.

***3.*** *Study the continuity and differentiability of the following function :*

$$f(x) = \begin{cases} x^3 \, sin\left(\frac{1}{x}\right) & if \; x \neq 0 \\ 0 & if \; x = 0. \end{cases}$$

**correction**

**1.** Using the intermidiate value theorem, verify that the equation $(x^3 + x^2 - 4x + 1 = 0)$ has at least one solution $x_0 \in ]0, 1[$. If we consider the function $f(x) = x^3 + x^2 - 4x + 1$, then :

f is a continuous function on $[0, 1]$ and $f(0) = 1$ and $f(1) = -1$, then $(f(0)f(1) < 0)$. 0.5pt+0.5pt

From the intermediate value theorem,we deduce that $(\exists x_0 \in ]0, 1[; f(x_0) = 0)$, this shows that the equation $(x^3 - 4x^2 + 6 = 0)$ has at least one solution $x_0 \in ]0, 1[$ 0.5pt+0.5pt

**2.** • We have $\lim\limits_{x \to 0} \dfrac{ln(1 + x^2)}{\sin^2(x)} = \dfrac{0}{0}$ it is an indeterminate form. If we set $f(x) = ln(1 + x^2)$ and $g(x) = \sin^2(x)$,

then $\lim\limits_{x \to 0} \dfrac{f'(x)}{g'(x)} = \lim\limits_{x \to 0} \dfrac{\frac{2x}{1+x^2}}{2\sin(x)\cos(x)} = \lim\limits_{x \to 0} \dfrac{x}{\sin(x)} \dfrac{1}{(1 + x^2)\cos(x)} = 1$, and using l'Hopital's Rule, we obtain :

$$\lim_{x\to 0}\frac{ln(1+x^2)}{ln(1+x^2)} = 1 \quad \text{1pt}$$

• We have $\lim_{x\to 0}\dfrac{arctan(x)}{x+x^2} = \dfrac{0}{0}$ it is an indeterminate form. If we set $f(x) = arctan(x)$ and $g(x) = x + x^2$, then $\lim_{x\to 0}\dfrac{f'(x)}{g'(x)} = \lim_{x\to 0}\dfrac{\frac{1}{1+x^2}}{1+2x} = 1$, and using l'Hopital's Rule, we obtain :$\lim_{x\to 0}\dfrac{arctan(x)}{x+x^2} = 1$ 1pt

**3.**

$$f(x) = \begin{cases} x^3 \ sin\ (\frac{1}{x}) & si\ x \neq 0 \\ 0 & si\ x = 0 \end{cases}$$

• The cotinuity 1pt

f is continuous on $\mathbb{R}^*$

Continuity at $x_0 = 0$ :

We calculate $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^3 \ sin\ (\dfrac{1}{x})$

We have : $-1 \leq sin(\frac{1}{x}) \leq 1 \implies \begin{cases} -x^3 \leq x^3 \ sin\ (\frac{1}{x}) \leq x^3 & if\ x > 0 \\ x^3 \leq x^3 \ sin\ (\frac{1}{x}) \leq -x^3 & if\ x < 0 \end{cases}$

By taking the limit we obtain $\lim_{x\to 0} x^3 \ sin\ (\dfrac{1}{x}) = 0$

Then f is continuous at 0. Consequently it is continuous on $\mathbb{R}$

• The differentiability 1pt

f is differntiable on $\mathbb{R}^*$

Differentiability at $x_0 = 0$ :

We calculate $\lim_{x\to 0}\dfrac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\dfrac{x^3 \ sin\ (\frac{1}{x}) - 0}{x} = \lim_{x\to 0} x^2 \ sin\ (\dfrac{1}{x})$.

We have $-1 \leq sin(\frac{1}{x}) \leq 1 \implies -x^2 \leq x^2 \ sin\ (\frac{1}{x}) \leq x^2$

By taking the limit we obtain $\lim_{x\to 0} x^2 \ sin\ (\dfrac{1}{x}) = 0$

Then f is differentiable at 0. Consequently it is differentiable on $\mathbb{R}$.