Concepts of random variable

Distributions of a random variable

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Random Variable: Introduction

- A random variable represents the quantity measured in a random experiment.
- Examples:
  - Age (in years)
  - Systolic blood pressure (in mmHg)
  - Disease severity stages (0-1-2-3-4)
  - Sex (Male/Female)
  - Eye color (brown, black, green, blue, . . .)
  - Cancer status (Present/Absent)
  - Number of patients

  $\rightarrow$ Quantities that vary from one individual to another.
- The possible outcomes of the experiment correspond to the possible values of the random variable.

Types of Random Variables

- Types of random variables:
  - If the outcomes are numerical (quantitative variable):
    - Continuous random variable: values span $\mathbb{R}$ or an interval
    - Discrete random variable: values are isolated (e.g., $\mathbb{N}$)
  - Otherwise (qualitative variable):
    - Ordinal random variable: values have a natural order
    - Nominal or categorical random variable: values without order

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Random Variable: Coding Example 1

- If Heads, A wins 10 DA.
- If Tails, A loses 10 DA.

- $\Omega = \{Heads, Tails\}$, with $\Pr(Heads) = \Pr(Tails) = 0.5$
- Let G be A's gain:
  - $G = +1$ if Heads
  - $G = -1$ if Tails
- $\Pr(G = +1) = \Pr(\text{"Heads"}) = 0.5$ and $\Pr(G = -1) = \Pr(\text{"Tails"}) = 0.5$
- Distribution of G: $\{(+1, 0.5), (-1, 0.5)\}$

G is a random variable that follows a specific probability distribution.

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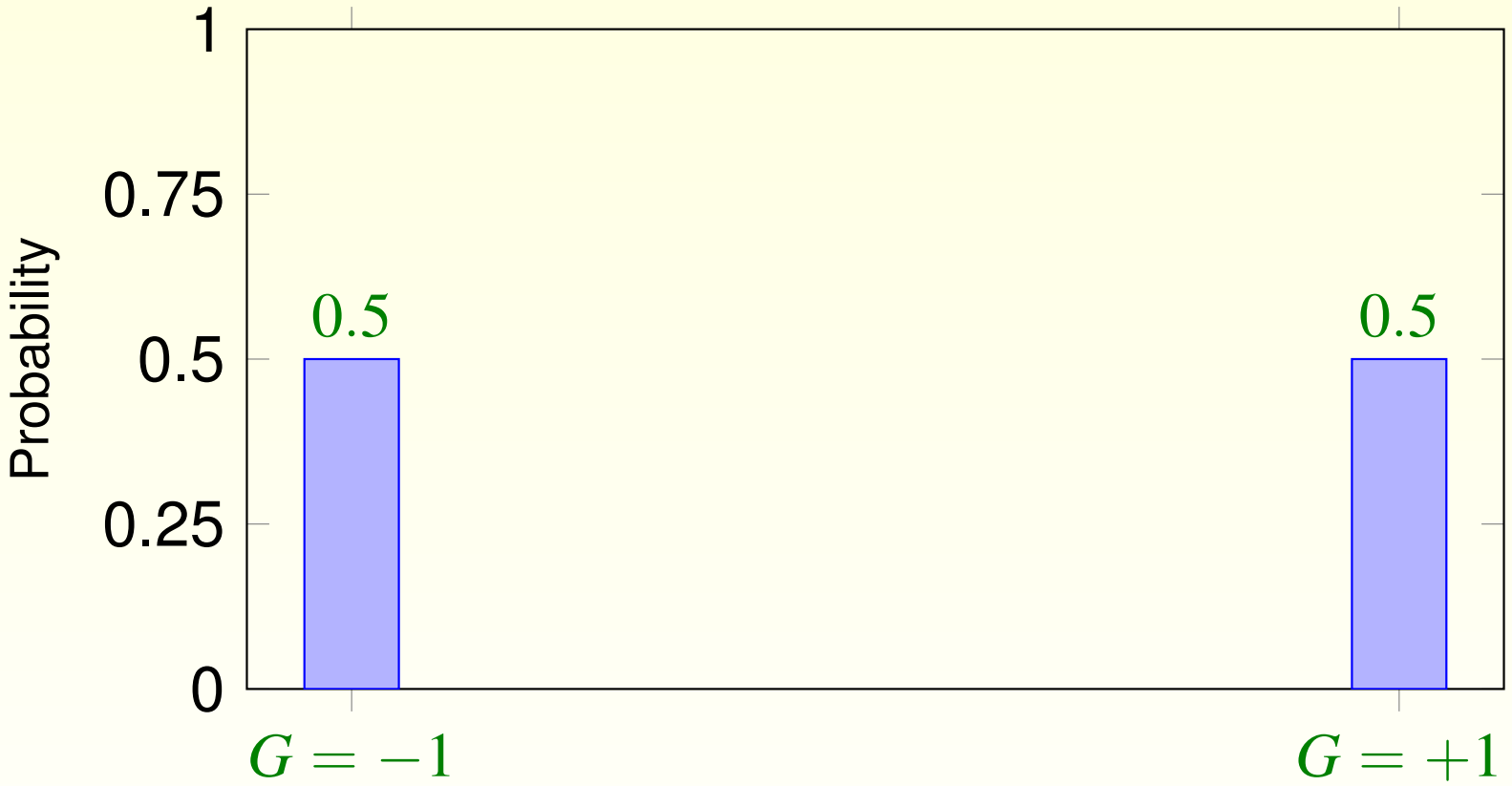
Distribution of the Random Variable G



Random Variable: Definition

- In the strict sense, a **random variable** always produces a **numerical outcome**.
- A random variable is a numerical coding of the outcomes of a random experiment.
  - If the outcomes are already numerical, the **coding** can simply be the identity.
- We will now revisit what we have learned about events and probabilities, and adapt these ideas to random variables.

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Discrete Random Variable: Definition

- Let $(\Omega, \mathcal{A})$ be a measurable space (an experiment equipped with a collection of admissible events).
- Let E be any finite or countably infinite set.
- Let $\mathbf{Pr}$ be a probability measure on $(\Omega, \mathcal{A})$.

Definition: A **discrete random variable** is any mapping X from Ω to a **finite or countable** subset E of $\mathbb{R}$:

$$\begin{array}{rcl} X : \Omega & \to & E \subset \mathbb{R} \\ \omega & \mapsto & X(\omega). \end{array}$$

satisfying: for every $x \in E$, the set $X^{-1}(\{x\})$ (will be denoted by $(X = x)$ or $\{X = x\}$) is an event.

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Random Variable: Example

- Dice experiment: $\Omega = \{f_1, f_2, f_3, f_4, f_5, f_6\}$.
- Define the random variable X by: $X(f_1) = 10$, $X(f_2) = 20$, $\ldots$, $X(f_6) = 60$.
- $[20 \leq X \leq 35] = \{f_2, f_3\}$.
- $[X < 35] = \{f_1, f_2, f_3\}$.
- $[X = 40] = \{f_4\}$.
- $[X = 35] = \varnothing$.

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Random Variable: Example 2

- If Ω is the set of students in a class, we may associate to each student ω the number $X(\omega)$ of his or her brothers and sisters. Then X takes values in a finite set: this is a discrete random variable.
- Sum of the results of two dice:

$$\begin{array}{ccc} X : \Omega & \longrightarrow & E \\ (m,n) & \mapsto & m+n. \end{array}$$

For example,

$$\{X = 3\} = X^{-1}(\{3\}) = \{(1,2),(2,1)\}.$$

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Partition Induced by a Discrete Random Variable

- Let $(\Omega, \mathcal{A})$ be a probability space and X a discrete (finite) random variable taking values in E.
- Since X is finite-valued, we can write $X(\Omega) = \{x_1, x_2, \ldots, x_n\}$.
- Then:

Theorem

The family of sets $\{\{X = x_i\},\ i = 1, \ldots, n\}$ forms a complete system of events (a partition of Ω).

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Law of a Discrete Random Variable

- Given $(\Omega, \mathbf{Pr})$ and a random variable $X : \Omega \to E$.
- Write $\Omega = \{\omega_1, \ldots, \omega_n\}$ and $E = \{x_1, \ldots, x_k\}$.
- Define

$$p_i = \mathbf{Pr}(X = x_i) = \mathbf{Pr}\big(\{\omega \in \Omega : X(\omega) = x_i\}\big).$$

- Observe that

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} \mathbf{Pr}(X = x_i) = \mathbf{Pr}\left(\bigcup_{i=1}^{k} \{\omega : X(\omega) = x_i\}\right) = \mathbf{Pr}(\Omega) = 1,$$

  because the sets $\{X = x_i\}$ are pairwise disjoint.
- Hence $(p_1, \ldots, p_k)$ defines a probability measure on E.

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Law of a Discrete Random Variable

Definition

The **distribution** (or **law**) of the discrete random variable X is the probability measure on E defined by

$$p_i = \mathbf{Pr}(X = x_i).$$

We denote this distribution by $\mathbf{Pr}_X$.

- This definition naturally extends to discrete random variables taking values in a countably infinite set.
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Discrete Random Variable: Example 1

- Two coins are tossed. The sample space contains 4 equally likely outcomes: PP, PF, FP, FF, each with probability $1/4$.
- Let X be the random variable counting the number of Heads (Pile).
- Then X takes the values $0, 1, 2$.
- We obtain:

$$\mathbf{Pr}(X = 0) = \frac{1}{4}, \qquad \mathbf{Pr}(X = 1) = \frac{1}{2}, \qquad \mathbf{Pr}(X = 2) = \frac{1}{4}.$$

- The probability distribution is often presented in a table:

X	0	1	2
$\mathbf{Pr}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Discrete Random Variable: Example 2

- Two dice are rolled successively.
- Let X be the random variable "sum of the two dice".
- Its distribution is defined on $\{2, 3, \ldots, 12\}$.

For example:

$$\mathbf{Pr}(X = 3) = \mathbf{Pr}(\{(1,2),(2,1)\}) = \frac{2}{36} = \frac{1}{18}.$$

X	2	3	4	5	6	7	8	9	10	11	12
$\mathbf{Pr}(X = x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Discrete Random Variable: Example 3

- Draw 4 balls from a bag containing 4 white balls and 6 red ones.
- Let X be the number of white balls obtained.
- Possible values: $0, 1, 2, 3, 4$.

Discrete Random Variable: Example 3

Let X be the number of white balls. Then:

$$X(\omega) \in \{0, 1, 2, 3, 4\}.$$

$$\mathbf{Pr}(X = x_i) = \frac{\text{favourable outcomes}}{\text{Total possible draws}}.$$

- Total possible draws: $C_{10}^4 = 210$.
- 0 white: $C_4^0 \times C_4^6 = 15$.
- 1 white: $C_4^1 \times C_3^6 = 80$.
- 2 white: $C_4^2 \times C_2^6 = \binom{4}{2}\binom{6}{2} = 90$.
- 3 white: $C_4^3 \times C_1^6 = \binom{4}{3}\binom{6}{1} = 24$.
- 4 white: $C_4^4 \times C_0^6 = \binom{4}{4}\binom{6}{0} = 1$.

Discrete Random Variable: Example 3-Distribution

X	0	1	2	3	4
$\mathbf{Pr}(X = x_i)$	$\frac{15}{210}$	$\frac{80}{210}$	$\frac{90}{210}$	$\frac{24}{210}$	$\frac{1}{210}$

We check:

$$\sum_{i=0}^{4} \mathbf{Pr}(X = x_i) = 1.$$

Random Variable: Coding Example

- Consider a disease M for which treatment must start before diagnostic confirmation.
- The treatment may cause adverse effects (AE).
- We know:
  - $\Pr(M^+) = 5\%$,
  - $\Pr(AE^+ \mid M^+) = 30\%$,
  - $\Pr(AE^- \mid M^-) = 85\%$.

Random Variable: Example (Cont.)

	M^+	M^-
AE^+	$\Pr(AE^+ \cap M^+) = 0.3 \times 0.05 = 1.5\%$	$\Pr(AE^+ \cap M^-) = 0.15 \times 0.95 = 14.25\%$
AE^-	$\Pr(AE^- \cap M^+) = 0.7 \times 0.05 = 3.5\%$	$\Pr(AE^- \cap M^-) = 0.85 \times 0.95 = 80.75\%$

- X is the indicator random variable for adverse effects (AE).
- $\mathbf{Pr}(X = 0) = \mathbf{Pr}(AE^-) = \mathbf{Pr}(AE^- \cap M^+) + \mathbf{Pr}(AE^- \cap M^-) = 0.84$;
- $\mathbf{Pr}(X = 1) = \mathbf{Pr}(AE^+) = \mathbf{Pr}(AE^+ \cap M^+) + \mathbf{Pr}(AE^+ \cap M^-) = 0.16$;
- Therefore, the distribution of X is: $\{(0, 0.84), (1, 0.16)\}$.

Plan de cours

Measures of Position: Mean or Expectation

- **Mean** at the population level
- Notation: $\mathbb{E}(X) = \mu_X = \mu$
- Computation: sum of all values weighted by their probabilities
  - Discrete variable X
    - Let X be a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, such that $\sum_{i=1}^{n} p_i = 1$:

$$\mu = \mathbb{E}(X) = \sum_{i=1}^{n} p_i x_i$$

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Measures of Position: Example

- Example 1: $\mu = (p \times 1) + (q \times 0) = p$
- Example 2: $\mu = (1 + 2 + 3 + 4 + 5 + 6)/6 = 3.5$

Measures of Dispersion: Variance and Standard Deviation

- **Variance** measures the spread around the mean
- Notation: $\text{var}(X) = \sigma_X^2 = \sigma^2$
- Definition: $\text{var}(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$
- Computation for discrete X:

$$\sigma^2 = \sum_{i=1}^{n} p_i [x_i - \mu]^2 = \sum_{i=1}^{n} p_i x_i^2 - (\mathbb{E}(X))^2$$

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Measures of Dispersion: Example

- Example 1: $\sigma^2 = p \cdot (1-p)^2 + q \cdot (0-p)^2 = pq$
- Example 2:
  $\sigma^2 = \frac{1}{6}\left[(1-3.5)^2 + (2-3.5)^2 + \ldots + (6-3.5)^2\right] = 2.9$

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CDF: Definition

- Let X be a discrete random variable on Ω with $X(\Omega) = \{x_1, x_2, \ldots, x_n\}$, $x_1 < x_2 < \ldots < x_n$.
- We want a function giving the probability for any interval $[a, b]$
- Consider the event $[X \leq x]$, i.e., all outcomes with values less than or equal to x.
- $\Pr(X \leq x)$ depends on x
- $F_X(x) = F(x) = \Pr(X \leq x)$ is called the **CDF of** X.

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- $F_X(x) = F(x) = \Pr(X \leq x)$ is called the **CDF of** X.

CDF: Definition

- Let X be a discrete random variable on Ω with $X(\Omega) = \{x_1, x_2, \ldots, x_n\}$, $x_1 < x_2 < \ldots < x_n$.
- We want a function giving the probability for any interval $[a, b]$
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CDF: Formal Definition

- **Definition:**

$F : x \in E \subset \mathbb{R} \mapsto [0, 1]$, defined by

$$F(x) = \mathbf{Pr}(X \leq x).$$

CDF: Properties

1. F is a non-decreasing step function;
2. $0 \leq F(x) \leq 1$;
3. If $x < x_1$, then $F(x) = 0$;
4. If $x_j \leq x < x_{j+1}$, then $F(x) = \sum_{i=1}^{j} p_i$;
5. If $x \geq x_n$, then $F(x) = 1$;
6. For countably infinite X: $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$;
7. F is discontinuous (right-continuous at each x_i);
8. At each jump point x, the jump height equals $p_j = \mathbf{Pr}(X = x_j)$;
9. For $a < b$: $\mathbf{Pr}(X \leq b) = \mathbf{Pr}(X \leq a) + \mathbf{Pr}(a < X \leq b)$ since the events are disjoint.

CDF: Remarks

- F allows calculation of probabilities between two values a and b:

$$\mathbf{Pr}(a \leq X \leq b) = F(b) - F(a),$$

(similar to cumulative frequency function)

- Knowing the CDF F of a discrete variable X, we can recover its probability distribution:

$$\mathbf{Pr}(X = x_j) = p_j = F(x_j) - F(x_{j-1}), \quad j \geq 2.$$

CDF of a Discrete Random Variable: Coin Toss

- Coin toss: $\Omega = \{H, T\}$
- Random variable: $X(T) = 0$, $X(H) = 1$
- Probabilities: $\Pr(X = 0) = 0.5$, $\Pr(X = 1) = 0.5$

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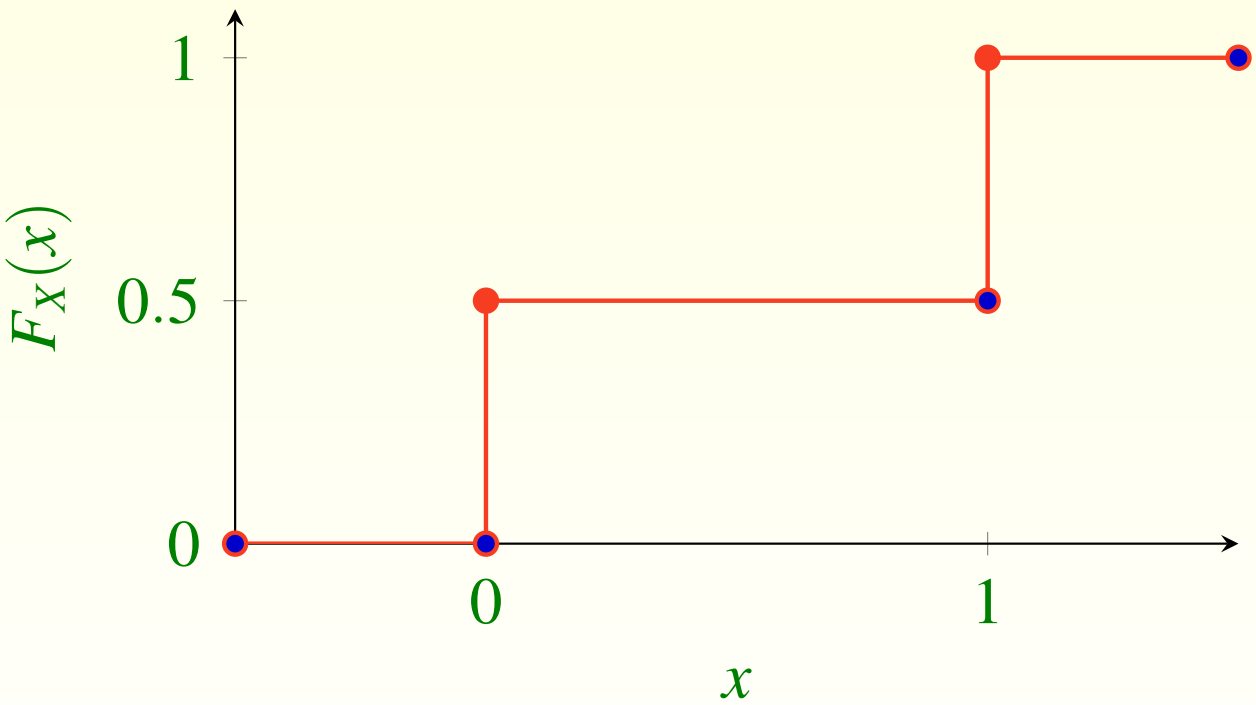

Plan de cours

Introduction

- A continuous random variable takes values in an uncountably infinite set.
- Examples:
  - Lifetime of a car battery
  - Arrival time of cars at a toll booth
- These examples cannot be modeled with discrete random variables.

Probability Density: Definition

Definition

A **probability density function** $f : \mathbb{R} \to \mathbb{R}$ satisfies:

- $f(x) \geq 0$ for all x;
- f is continuous on $\mathbb{R}$ except possibly at a finite number of points;
- $\displaystyle\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

Continuous Random Variable: Definition

A real random variable X is called **continuous** if there exists a measurable, non-negative function f_X such that

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

and for any subset $A \subset \mathbb{R}$,

$$\Pr(X \in A) = \int_A f_X(x)\,dx.$$

- f_X is called the **density of** X, and X is a continuous random variable.

Example 1: Uniform Density

- Let X be the angle between a puncture and the valve on a wheel. Its density is:

$$f(x) = \frac{1}{2\pi}, \quad x \in [0, 2\pi].$$

- Then X follows the **uniform distribution** on $[0, 2\pi]$.

Example 2: Normal Density

- Let X represent students' heights in a class. Then X is continuous and often modeled by:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(- \frac{(x-\mu)^2}{2\sigma^2} \right), \quad x \in \mathbb{R}.$$

- X follows the **Normal distribution** with mean μ and standard deviation σ.

Plan de cours

Cumulative Distribution Function (CDF)

- The CDF of X is defined as:

$$F(a) = \Pr(X < a) = \int_{-\infty}^{a} f(x)\,dx.$$

- For any $a, b \in \mathbb{R}$:

$$\Pr(a \leq X < b) = F(b) - F(a).$$

- It follows that:

$$\Pr(X = x) = 0, \quad \Pr(x \leq X < x + dx) = f(x)\,dx.$$

CDF: Properties

1. F is non-decreasing;
2. $0 \leq F(x) \leq 1$;
3. $\lim_{x \to -\infty} F(x) = 0, \quad \lim_{x \to +\infty} F(x) = 1$;
4. F is right-continuous;
5. For $a < b$, $\Pr(X \leq b) = \Pr(X \leq a) + \Pr(a < X \leq b)$ (disjoint events).

CDF: Theorem

Important Theorem

The CDF uniquely characterizes the distribution: two random variables have the same distribution if and only if they have the same CDF.

Plan de cours

Distribution of a Continuous Random Variable

- Probability density $f(x)$
- Probability of interval:
  $\Pr(a \leq X \leq b) = \int_a^b f(x)\,dx = F(b) - F(a)$
- Properties:
  - $f(x) \geq 0$
  - $\int_{-\infty}^{+\infty} f(x)\,dx = 1$
  - CDF: $F(x) = \int_{-\infty}^{x} f(t)\,dt$

Plan de cours

Expectation / Mean

- **Mean** of the population: $\mathbb{E}(X) = \mu_X = \mu$
- Formula:

$$\mu = \mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

- Expectation may not exist if the integral is undefined.
- Interpretation:
  - Point estimate if f known
  - Theoretical expectation if f unknown

Properties of Expectation

- Let X, Y be random variables and a, c constants:
  - $\mathbb{E}(c) = c$
  - $\mathbb{E}(X + c) = \mathbb{E}(X) + c$
  - $\mathbb{E}(X - \mathbb{E}(X)) = 0$ (centered variable)
  - $\mathbb{E}(aX) = a\mathbb{E}(X)$
  - $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Variance and Standard Deviation

- **Variance** measures spread around the mean
- Notation: $Var(X) = \sigma_X^2 = \sigma^2$
- Definition:

$$Var(X) = \mathbb{E}[(X - \mathbb{E}(X))^2] = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$$

- Calculation:

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mathbb{E}(X))^2 f(x)\, dx = \int_{-\infty}^{\infty} x^2 f(x)\, dx - (\mathbb{E}(X))^2$$

- σ^2 = Variance, σ = Standard deviation

Variance: Properties

- $Var(X) \geq 0$; zero if X is constant
- Units:
  - Mean and σ have same unit as X
  - Variance has squared unit
- For constants c:
  - $Var(c) = 0$, $Var(X + c) = Var(X)$, $Var(cX) = c^2 Var(X)$
- $Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$
- Covariance: $Cov(X, Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$

Medicals Examples

Table: Examples of Continuous Random Variables in Medicine (Functional Data)

Variable	**Measurement / Unit**	**Functional Context (CFD)**
Blood pressure	mmHg	Continuous monitoring over 24h; yields a curve per patient
Heart rate	beats/min	Continuous ECG recording; heart rate variability curve
Body temperature	℃	Continuous temperature measurements over hospital stay
Glucose concentration	mg/dL	Continuous glucose monitoring (CGM) in diabetics; time series
Respiration rate	breaths/min	Continuous monitoring in ICU; respiratory dynamics curve
Oxygen saturation (SpO_2)	%	Continuous pulse oximeter data; functional trajectory over time
Drug concentration in blood	mg/L	Pharmacokinetic study; concentration-time curve
Electroencephalogram (EEG) signal	μV	Continuous recording of brain activity; functional signal over tim

Key Remarks: The Role of Probability Distributions

- **Definition and Modeling:** Probability distributions (or probability laws) are mathematical functions ($P(x)$, $f(x)$, $F(x)$) that describe the likelihood of observing **all possible values** for a theoretical random variable ($\mathbf{X}$). They serve as the model for a random phenomenon.
- **Direct Probability Calculation:** Their primary use is the direct calculation of the probability of specific events.
  - **Discrete Example:** Calculating $P(X = k)$ for the Binomial distribution.
  - **Continuous Example:** Calculating $P(a \leq X \leq b)$ for the Normal distribution.

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Key Remarks: The Role of Probability Distributions

- **Statistical Inference and Testing:** During parametric statistical tests (like Student's t-test or ANOVA), we reference these distributions (e.g., t, χ^2, F distributions) to determine the p-value: the probability that the observed data is consistent with the null hypothesis (H_0).

$\Rightarrow$ **Prerequisite:** A deep understanding of probability distributions requires proficiency in **Probability Theory** (sample space, events, conditional probability, etc.).

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Summary and Details

Feature	Discrete Random Variable (R.V.)	Continuous Random Variable (R.V.)
Example of Values	Countable (e.g., $\{0, 1, 2, \dots\}$)	Uncountable (Interval: $[a, b]$)
Definition of the Probability Law	**Probability Mass Function** (PMF) $p_i = \mathbf{P}(X = x_i)$	**Probability Density Function** (PDF) $f(x)$
Interval Calculation	$\mathbf{P}([a \leq X \leq b]) = \sum_{a \leq x_i \leq b} p_i$	$\mathbf{P}([a \leq X \leq b]) = \int_a^b f(x)dx$ $= F(b) - F(a)$
Fundamental Properties (Total Sum/Integral)	$p_i \geq 0$ $\sum_{i=1}^{n} p_i = 1$	$f(x) \geq 0$ $\int_{-\infty}^{\infty} f(x)dx = 1$
Cumulative Distribution (CDF) $F(x)$	Step function (jumps) $F(x) = \sum_{x_i \leq x} p_i$	Smooth, continuous function $F(x) = \int_{-\infty}^{x} f(t)dt$
Probability of a Single Point	$\mathbf{P}(X = x)$ is p_i	$\mathbf{P}(X = x) = 0$ (single point)
Interpretation of $f(x)$	N/A	$f(x)$ is not a probability. $f(x)dx \approx \mathbf{P}(x \leq X \leq x + dx)$

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