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Mechanics of the Material Point

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PREFACE

This book is designed as a teaching aid intended primarily for first-year students in the Science and Technology (S.T.) program at the Faculty of Technology of Djillali Liabes University in Sidi Bel Abbes. It covers the content of the Material Point Mechanics module, which presents the fundamental principles governing the motion of objects and the laws that describe their mechanical behavior.

The program includes the study of the kinematics and dynamics of material points, Newton's laws of motion, the concepts of work and energy, the principles of conservation, and the analysis of forces acting on a system. Emphasis is placed on acquiring a solid understanding of these fundamental concepts, which are essential in all fields of engineering and applied sciences.

The aim of this book is to provide a clear and structured reference where theoretical concepts are presented through concise explanations. This approach aims to facilitate understanding, strengthen problem-solving skills, and help students acquire the analytical skills necessary for more advanced studies in mechanics and technology.

We hope that this book will not only be a valuable companion during lectures and tutorials, but also a practical guide for exam preparation. It may also be useful for students who wish to deepen their knowledge of classical mechanics and strengthen their scientific foundation for future academic and professional pursuits.

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INTRODUCTION

The Mechanics of the Material Point module introduces the fundamental principles governing the motion of objects and the forces that affect them. As one of the essential pillars of classical physics, this subject forms the foundation for many advanced disciplines in engineering, technology, and applied sciences.

In this module, the material point is considered as an idealized object whose dimensions are negligible compared to the distances involved. This simplification allows the study of motion in a precise and systematic way while developing rigorous analytical skills. The concepts covered include the description of motion in space (kinematics), the laws that govern this motion (dynamics), and the fundamental principles of work, energy, momentum, and their conservation.

The study of the mechanics of the material point aims to provide students with the tools necessary to model real physical systems, analyze their behavior, and solve practical problems encountered in engineering. It also promotes the development of logical reasoning, mathematical formulation, and scientific methodology.

This module represents a crucial step in the scientific training of first-year students in Sciences and Technologies, preparing them for more advanced topics such as rigid body mechanics, fluid mechanics, electromagnetism, and system dynamics. Through theoretical explanations, solved exercises, and application problems, students will progressively acquire a solid understanding of the principles governing motion and forces.

Chapter 1

Mathematical tools

1.1 Scalar quantities and vector quantities

Physical quantities can be scalar or vector in nature.

1.1.1 Scalar quantities

Scalar physical quantities are defined entirely by a number and an appropriate unit. Examples include: temperature, mass, pressure at a point, energy of a system, etc.

1.1.2 Vector quantities

A vector physical quantity is a quantity specified by a number and an appropriate unit plus a direction and a sense. Geometrically, it is represented by a vector having the same direction, the same sense, and a magnitude measured by choosing a corresponding graphic unit, i.e., the scale. Examples of vector quantities include velocity, weight, electric and magnetic fields, etc.

Example: The weight of a body with a mass of $1kg$ can be represented by a vector with the following characteristics:

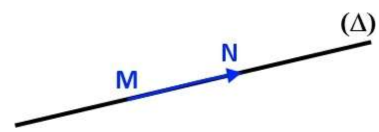
- *origin:* the center of gravity of the object;
- *direction:* vertical;
- *direction:* from top to bottom;
- *module:* the weight being $9.8N$, if we choose a scale where $1cm$ corresponds to $2N$ ($1\text{cm} \rightarrow 2N$), the vector will have a length of $4.9cm$.

1.2 Vectors

1.2.1 Definition

A vector $\overrightarrow{MN}$ is a segment with origin M and endpoint N. It is defined by:

- Its origin or point of application M.
- The direction, which is that of the line (Δ).
- Its direction.
- Its magnitude (length MN).



1.2.2 Different types of vectors

There are different types of vectors:

- Fixed vector: fixed origin, support, modulus, and direction.
- Sliding vector: fixed support, modulus, and direction.
- Free vector: fixed modulus and direction.

1.2.3 Modulus of the vector

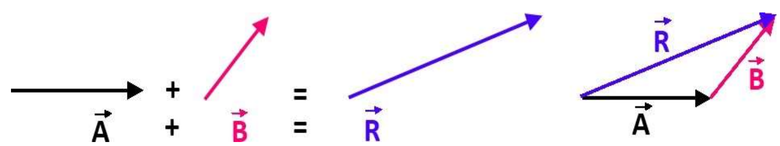
A unit of length having been chosen on the line (Δ), support of the vector $\overrightarrow{MN}$, we call the modulus of the vector $\overrightarrow{MN}$, denoted by $|\overrightarrow{MN}|$, the length MN.

Special case: if $|\overrightarrow{MN}| = 1$, the vector is said to be unit. It can be used to measure any vector that is parallel to it.

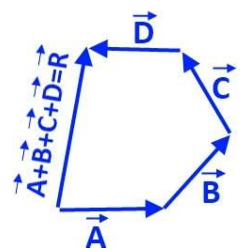
1.3 Basic operations on vectors

1.3.1 Vector addition

Geometrically, the addition of two vectors is performed by merging the origin of the second vector with the endpoint of the first vector. The vector with the origin of the first vector as its origin and the endpoint of the second vector as its endpoint defines the sum of the two vectors.



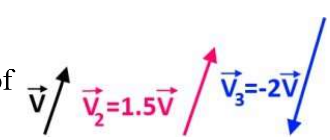
The method can be generalized for the addition of more than two vectors.



1.3.2 Product of a vector and a scalar

Definition

The product of a vector $\vec{V}$ by a scalar α is a vector $\alpha\vec{V}$ having the same direction as the vector $\vec{V}$, with modulus $|\alpha||\vec{V}|$ and whose direction is that of $\vec{V}$ if $\alpha > 0$ and opposite to that of $\vec{V}$ if $\alpha < 0$.



Properties

Multiplying a vector by a scalar satisfies the following properties:

- Distributivity with respect to vector addition:

$$\alpha(\vec{U} + \vec{V}) = \alpha\vec{U} + \alpha\vec{V} \quad (1.1)$$

- Distributivity with respect to scalar addition:

$$(\alpha + \beta)\vec{U} = \alpha\vec{U} + \beta\vec{U} \quad (1.2)$$

- Associativity:

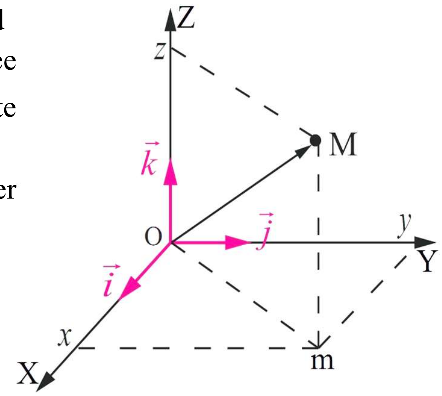
$$\alpha(\beta\vec{U}) = (\alpha\beta)\vec{U} \quad (1.3)$$

1.3.3 Components of a vector

In many physical situations, it is important to use a coordinate system as a reference system. The reference point *R* (*O*; *X, Y, Z*) consists of an origin point *O* and a system of three axes (*OX*), (*OY*), and (*OZ*) defining the three dimensions of space. An orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$ to this coordinate system.

It is a basis consisting of vectors that are orthogonal to each other and unitary:

- $\vec{\imath}$: Unit vector of the axis (*OX*).
- $\vec{\jmath}$: Unit vector of the axis (*OY*).
- $\vec{k}$: Unit vector of the axis (*OZ*).



The components (*x, y, z*) of a vector $\vec{V} = \overrightarrow{OM}$ are the orthogonal projections of the vector position on the three axes of the coordinate system. In this case, the position vector is written as:

$$\vec{V} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}_{(\vec{\imath},\vec{\jmath},\vec{k})} \quad (1.4)$$

1.3.4 Equality of two vectors

Two vectors $\vec{V}_1 = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\vec{V}_2 = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$ are equal i f their components are equal one-to-one; i.e., $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$.

1.3.5 Scalar product of two vectors

Definition

The scalar product of two vectors $\vec{V}_1$ and $\vec{V}_2$ is the **scalar** denoted $\vec{V}_1.\vec{V}_2$ and defined by:

$$\vec{V}_1.\vec{V}_2 = \|\vec{V}_1\|\,\|\vec{V}_2\| \cos\left[\text{angle}\left(\vec{V}_1, \vec{V}_2\right)\right] \quad (1.5)$$

If (x_1 , y_1 , z_1) and (x_2 , y_2 , z_2) are the respective components of vectors V_1 and V_2 in ***the same orthonormal basis***, we also have:

$$\vec{V}_1.\vec{V}_2 = x_1x_2 + y_1y_2 + z_1z_2 \quad (1.6)$$

Properties

- The scalar product is commutative:

$$\vec{U}.\vec{V} = \vec{V}.\vec{U} \quad (1.7)$$

- The scalar product is distributive with respect to addition:

$$\vec{U}.\left[\vec{V} + \vec{W}\right] = \vec{U}.\vec{V} + \vec{U}.\vec{W} \quad (1.8)$$

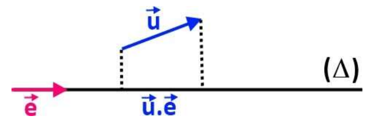
- The scalar product allows us to define the modulus of a vector:

$$\|\vec{V}\| = \sqrt{\vec{V}.\vec{V}} = \sqrt{\vec{V}^2} = \sqrt{x^2 + y^2 + z^2} \tag{1.9}$$

- $\vec{U} \neq 0$ and $\vec{V} \neq 0$, if $\vec{U}.\vec{V} = 0$ then $\vec{U}$ and $\vec{V}$ are orthogonal $(\vec{U} \perp \vec{V})$

Applications

The projection of any vector $\vec{U}$ onto an oriented axis (Δ) of unit vector $\vec{e}$ is given by:



$$\text{proj}_{\Delta}\, \vec{U} = \vec{U}.\vec{e}$$

The cosine of the angle between two vectors $\vec{V}_1$ and $\vec{V}_2$ with respective components (x_1 , y_1 , z_1) and (x_2 , y_2 , z_2) in the same basis is:

$$\cos\left[\text{angle }\left(\vec{V}_1.\vec{V}_2\right)\right] = \frac{\vec{V}_1.\vec{V}_2}{\|\vec{V}_1\|\ \|\vec{V}_2\|} = \frac{x_1x_2 + y_1y_2 + z_1z_2}{\sqrt{x_1^2 + y_1^2 + z_1^2}\,\sqrt{x_2^2 + y_2^2 + z_2^2}} \tag{1.10}$$

1.3.6 Vector product of two vectors

Definition

The vector product of two vectors $\vec{V}_1 = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k}$ and $\vec{V}_2 = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k}$ is the **vector** denoted $\vec{V}_1 \wedge \vec{V}_2$ such that:

- its direction is perpendicular to $\vec{V}_1$ and to $\vec{V}_2$ (and therefore to the plane generated by these two vectors when they are non-zero and non-collinear),
- its point of application is not fixed; it is a sliding vector,
- its meaning is such that the trihedron $(\vec{V}_1, \vec{V}_2, \vec{V}_1 \wedge \vec{V}_2)$ is direct (corkscrew rule),
- its norm is:

$$\|\vec{V}_1 \wedge \vec{V}_2\| = \|\vec{V}_1\|\ \|\vec{V}_2\| \sin\left[\text{angle }\left(\vec{V}_1, \vec{V}_2\right)\right] \tag{1.11}$$

and represents the area of the parallelogram constructed on vectors $\vec{V}_1$ and to $\vec{V}_2$.

- In terms of vector components, the vector product is expressed as follows:

$$\vec{V}_1 \wedge \vec{V}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} \wedge \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix} = \begin{pmatrix} y_1z_2 - y_2z_1 \\ x_2z_1 - x_1z_2 \\ x_1y_2 - x_2y_1 \end{pmatrix} \tag{1.12}$$

Properties

- The vector product is anticommutative:

$$\vec{U} \wedge \vec{V} = -\vec{V} \wedge \vec{U} \tag{1.13}$$

- The vector product is distributive with respect to addition:

$$\vec{U} \wedge [\vec{V} + \vec{W}] = \vec{U} \wedge \vec{V} + \vec{U} \wedge \vec{W} \tag{1.14}$$

- $\vec{U} \neq 0$ and $\vec{V} \neq 0$, if $\vec{U} \wedge \vec{V} = 0$ then $\vec{U}$ and $\vec{V}$ are collinear $(\vec{U} // \vec{V})$.

Double vector product, mixed product
For any vectors $\vec{U}, \vec{V}$ and $\vec{W}$, we have:

- $[\vec{U} \wedge \vec{V}] \wedge \vec{W} = [\vec{U}.\vec{W}]\vec{V} - [\vec{V}.\vec{W}]\vec{U}$
- $\vec{U} \wedge [\vec{V} \wedge \vec{W}] = [\vec{U}.\vec{W}]\vec{V} - [\vec{U}.\vec{V}]\vec{W}$
- $[\vec{U} \wedge \vec{V}].\vec{W} = [\vec{V} \wedge \vec{W}].\vec{U} = [\vec{W} \wedge \vec{U}].\vec{V}$

The mixed product $(\vec{U}, \vec{V}, \vec{W})$ defined above represents the volume of the parallelepiped constructed on the vectors $\vec{U}, \vec{V}$ and $\vec{W}$.

Applications

A basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is direct if:

$$\vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \qquad \vec{e}_2 \wedge \vec{e}_3 = \vec{e}_1 \quad \text{and} \quad \vec{e}_3 \wedge \vec{e}_1 = \vec{e}_2 \tag{1.15}$$

with $\|\vec{e}_1\| = \|\vec{e}_2\| = \|\vec{e}_3\| = 1$

1.3.7 Derivative of a vector

Definition

The derivative of the vector $\vec{V}(t) = x(t)\,\vec{\imath} + y(t)\,\vec{\jmath} + z(t)\,\vec{k}$ in the fixed base $(\vec{\imath}, \vec{\jmath}, \vec{k})$ whose components are the derivatives of the components of the vector $\vec{V}(t)$:

$$\frac{d\vec{V}(t)}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} \tag{1.16}$$

It is important to note that in this case the basis vectors are considered fixed; i.e.:

$$\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = \vec{0} \tag{1.17}$$

Properties

- Linearity:

$$\frac{d(\alpha\vec{V}_1 + \beta\vec{V}_2)}{dt} = \alpha\frac{d\vec{V}_1}{dt} + \beta\frac{d\vec{V}_2}{dt} \tag{1.18}$$

- Derivative of a scalar product:

$$\frac{d(\vec{V}_1.\vec{V}_2)}{dt} = \frac{d\vec{V}_1}{dt}.\vec{V}_2 + \vec{V}_1.\frac{d\vec{V}_2}{dt} \tag{1.19}$$

- Derivative of a vector product:

$$\frac{d(\vec{V}_1 \wedge \vec{V}_2)}{dt} = \frac{d\vec{V}_1}{dt} \wedge \vec{V}_2 + \vec{V}_1 \wedge \frac{d\vec{V}_2}{dt} \tag{1.20}$$

- Derivative of the product of a vector by a scalar function:

$$\frac{d(f\vec{V})}{dt} = \frac{df}{dt}\vec{V} + f\frac{d\vec{V}}{dt} \tag{1.21}$$

- It can also be shown that a vector with a fixed modulus $\|\vec{V}\| = V =$ Constant is orthogonal to its derivative $\left(\vec{V} \perp \frac{d\vec{V}}{dt}\right)$.

Proof:
We have:

$$\frac{d(\vec{V}^2)}{dt} = \frac{d(\vec{V}.\vec{V})}{dt} = 2\vec{V}.\frac{d\vec{V}}{dt} \tag{1.22}$$

on the other hand, we have:

$$\frac{d(\vec{V}^2)}{dt} = \frac{d(V^2)}{dt} = 2V.\frac{dV}{dt} = 0 \tag{1.23}$$

The last equality comes from the fact that the modulus is constant, so its derivative is zero. Comparing equations (1.22) and (1.23), we find that $\vec{V}.\frac{d\vec{V}}{dt} = \vec{0}$, which implies that the two vectors $\vec{V}$ and $\frac{d\vec{V}}{dt}$ are orthogonal.

1.4 Differential

1.4.1 Differential of a scalar function

Partial derivative of a function with multiple variables

Let $f(x, y, z)$ be a function of three variables. The partial derivative of $f(x, y, z)$ with respect to one of the variables is obtained by calculating the derivative while keeping the other two variables constant. Thus:

- the partial derivative of f with respect to x, denoted $\frac{\partial f}{\partial x}$ is obtained by differentiating with respect to x and while considering y and z as constants.
- the partial derivative of f with respect to y, denoted $\frac{\partial f}{\partial y}$ is obtained by differentiating with respect to y and while considering x and z as constants.
- the partial derivative of f with respect to z, denoted $\frac{\partial f}{\partial z}$ is obtained by differentiating with respect to z and while considering x and y as constants.

Example :

$$f(x,y,z) = xy^2 + \cos z \Longrightarrow \begin{cases} \dfrac{\partial f}{\partial x} = y^2 \\ \dfrac{\partial f}{\partial y} = 2xy \\ \dfrac{\partial f}{\partial z} = -\sin z \end{cases}$$

Total differential

The differential of the scalar field $f(x,y,z)$ is defined by:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad (1.24)$$

Geometrically, it represents the variation of the function f from a point $M(x, y, z)$ to an infinitely close point $M'(x+dx, y+dy, z+dz)$.

Example :

$$f(x,y,z) = xy^2 + \cos z \Longrightarrow df = y^2dx + 2xydy - \sin z dz$$

1.4.2 Differential of a vector function

The differential of a vector field $\vec{V}(x,y,z)$ is defined by:

$$d\vec{V} = \frac{\partial \vec{V}}{\partial x}dx + \frac{\partial \vec{V}}{\partial y}dy + \frac{\partial \vec{V}}{\partial z}dz \quad (1.25)$$

Geometrically, this represents the variation of the vector field $\vec{V}(x,y,z) = \overrightarrow{OM}$, when the material point moves from point $M(x, y, z)$ to the neighboring point $M'(x+dx, y+dy, z+dz)$, i.e. its variation $\overrightarrow{MM'}$.

1.5 Differential operators

1.5.1 Gradient

The gradient of a scalar function $f(x, y, z)$ is the vector denoted $\overrightarrow{\text{grad}}f$ and defined by the following formula:

$$\overrightarrow{\text{grad}}f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k} \quad (1.26)$$

It is convenient to introduce the differential operator $\vec{\nabla}$ (nabla) defined by:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \quad (1.27)$$

This allows us to write the gradient of a scalar function $f(x, y, z)$ in the following form:

$$\overrightarrow{\text{grad}}f = \vec{\nabla} f \quad (1.28)$$

1.5.2 Divergence

The divergence of a vector is $\vec{U}$ of components (U_x, U_y, U_z) is a scalar denoted div $\vec{U}$ by and defined by:

$$\text{div}\,\vec{U} = \vec{\nabla}.\vec{U} = \frac{\partial U_x}{\partial x} + \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial z} \quad (1.29)$$

1.5.3 Rotational

The rotational vector of a vector $\vec{U}$ with components (U_x, U_y, U_z) is a vector denoted $\overrightarrow{rot}\,\vec{U}$ and defined using the operator $\vec{\nabla}$:

$$\overrightarrow{rot}\,\vec{U} = \vec{\nabla} \wedge \vec{U} \tag{1.30}$$

the components of this vector are therefore:

$$\overrightarrow{rot}\,\vec{U} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \wedge \begin{pmatrix} U_x \\ U_y \\ U_z \end{pmatrix} = \begin{pmatrix} \frac{\partial U_z}{\partial y} - \frac{\partial U_y}{\partial z} \\ \frac{\partial U_x}{\partial z} - \frac{\partial U_z}{\partial x} \\ \frac{\partial U_y}{\partial x} - \frac{\partial U_x}{\partial y} \end{pmatrix} \tag{1.31}$$

Chapter 2

Coordinate Systems

In many problems, it is important to specify the orientation of a vector relative to an arbitrarily chosen reference. The reference directions, or axes of the reference system, will allow us to qualify the orientation of the vector relative to this system. The reference system that we choose is called a coordinate system. It consists of a system of axes and an origin. In physical space, one of the most commonly used coordinate systems is the Cartesian coordinate system. It consists of an origin O and three axes X, Y and Z. In the chosen coordinate system, a basis is then defined. In physics, we will exclusively use an orthonormal basis, i.e., a basis in which the three basis vectors are orthogonal to each other and unit vectors.

The position of the basis in the reference frame defines the coordinate system of the point in the reference frame. A widely used coordinate system is the Cartesian coordinate system. There are others, such as the polar, cylindrical, or spherical coordinate systems.

2.1 Cartesian coordinate systems

2.1.1 Definition

In the Cartesian coordinate system shown in Figure 2.1, the direction of the basis vectors $(\vec{i},\vec{j},\vec{k})$ of the coordinate system *R* (*O, X, Y, Z*) coincides with that of the axes of the coordinate system.
The vectors are orthonormal, i.e., orthogonal to each other and unit vectors (the length of the vector is equal to 1).

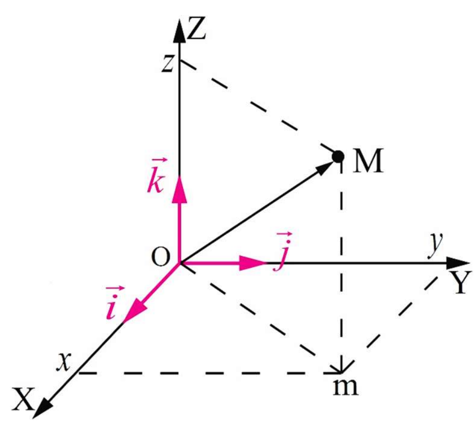


Figure 2.1 – Representation of the Cartesian coordinate system

Any point *M* in space is defined by its three coordinates (*x, y, z*) as follows:

$$x = \text{abscissa of } M, \qquad y = \text{ordinate of } M, \qquad z = \text{side of } M,$$
$$x = \text{Proj}_{\overrightarrow{OX}}\overrightarrow{OM}, \qquad y = \text{Proj}_{\overrightarrow{OY}}\overrightarrow{OM}, \qquad z = \text{Proj}_{\overrightarrow{Oz}}\overrightarrow{OM},$$

In the coordinate system R (O, X, Y, Z), the position vector $\overrightarrow{OM}$ is written as:

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} \tag{2.1}$$

2.1.2 Elementary displacement

Let M' be another point in space very close to point M. The elementary displacement vector $\overrightarrow{MM'}$ is given by:

$$\overrightarrow{MM'} = \overrightarrow{OM'} - \overrightarrow{OM} = d\overrightarrow{OM} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k} = \tag{2.2}$$

because in the coordinate system R (O, X, Y, Z) we have:

$$d\,\vec{\imath} = d\,\vec{\jmath} = d\,\vec{k} = \vec{0} \tag{2.3}$$

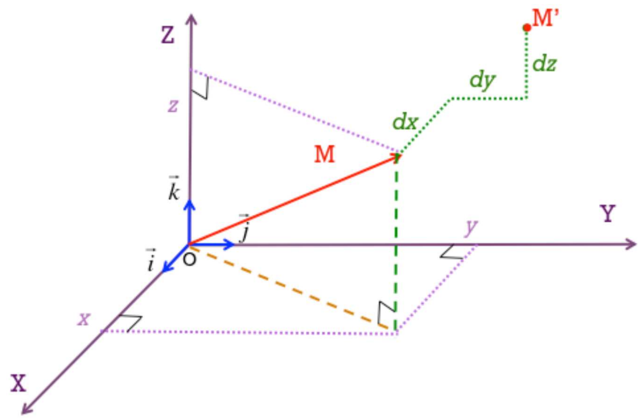


Figure 2.2 – Elementary displacement in Cartesian coordinates.

2.1.3 Elementary volume

The volume defined by the elementary displacement is called a volume element or elementary volume dV. In Cartesian coordinates, the elementary volume is a cube with dimensions dx, dy, and dz (see Figure 2.3):

$$dV = dxdydz \tag{2.4}$$

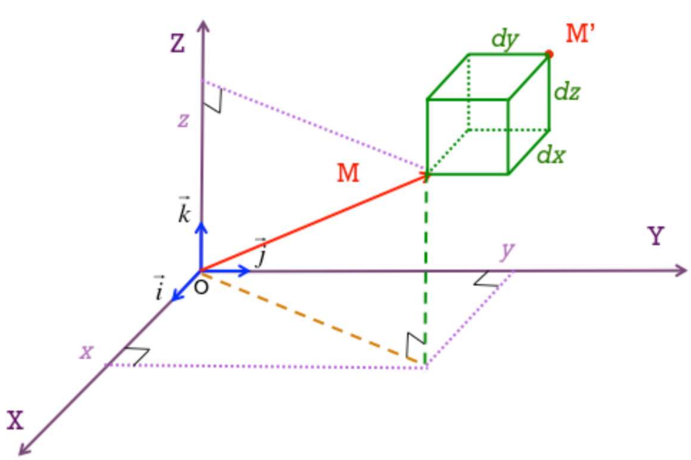


Figure 2.3 – Elementary volume in Cartesian coordinates.

2.2 Polar coordinate system

2.2.1 Definition

It is a coordinate system used to locate the position of a point M in two dimensions. Thus, the position of point M is located by the distance ρ, which separates it from the origin O, and by the angle φ that the vector $\overrightarrow{OM}$ makes with the axis (OX) (see figure (2.4-a))

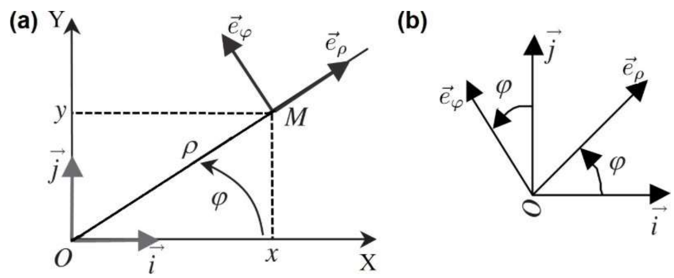


Figure 2.4 – (a) Representation of the polar coordinate system and (b) the associated basis ($\vec{e}_\varphi$, $\vec{e}_\rho$,).

- The origin point O corresponds to the pole, hence the term polar coordinate. The length of the segment OM corresponds to its radial coordinate. It is denoted by ρ (rho: Greek letter).
- The other coordinate is the angular coordinate, also called the polar angle and denoted *by* φ (phi: Greek letter). This angle is measured relative to the x-axis (OX), which is then called the polar axis.

Unlike the Cartesian coordinates x and y, the polar coordinates ρ and ϕ are not of the same nature. The radial coordinate ρ has the dimension of a length, like x and y. The angular coordinate is expressed in radians, which is a dimensionless unit of angle, so we have:

$$\rho = \left\|\overrightarrow{OM}\right\| \quad ; \quad 0 \leq \rho \leq +\infty$$

$$\varphi = \left(\widehat{\overrightarrow{OM}, \vec{\imath}}\right) \quad ; \quad 0 \leq \varphi \leq 2\pi$$

Using the diagram in Figure (2.4-a), we can find the relationships between Cartesian coordinates and polar coordinates:

$$\begin{cases} \cos\varphi = \dfrac{x}{\rho} \Longrightarrow x = \rho\cos\varphi \\ \sin\varphi = \dfrac{y}{\rho} \Longrightarrow y = \rho\sin\varphi \end{cases} \tag{2.5}$$

or even:

$$\begin{cases} \tan\varphi = \dfrac{\sin\varphi}{\cos\varphi} = \dfrac{y}{x} \Longrightarrow \varphi = \arctan\dfrac{y}{x} \\ \left\|\overrightarrow{OM}\right\| = \rho = \sqrt{x^2 + y^2} \end{cases} \tag{2.6}$$

To express the position vector $\overrightarrow{OM}$, it is convenient to introduce a new orthonormal basis $(\vec{e}_\rho, \vec{e}_\varphi)$ naturally associated with this coordinate system and defined as follows (see Figure (2.4-b)):

- $\vec{e}_\rho$ is the unit vector following the direction and sense of O towards M. This is the radial vector (following the radius).

- $\vec{e}_\varphi$ is the unit vector perpendicular to the vector $\vec{e}_\rho$. It is obtained by performing a rotation of an angle of $+\frac{\pi}{2}$ from the vector $\vec{e}_\rho$. It is the orthoradial vector (perpendicular to the radius). Using Figure (2.4-b), the components of the unit vectors $\vec{e}_\rho$ and $\vec{e}_\varphi$ in the Cartesian basis are:

$$\begin{cases} \vec{e}_\rho = \cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath} \\ \vec{e}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \end{cases} \tag{2.7}$$

with:

$$\begin{cases} \dfrac{d\vec{e}_\rho}{d\varphi} = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} = \vec{e}_\varphi \\ \dfrac{d\vec{e}_\varphi}{d\varphi} = -(\cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath}) = -\vec{e}_\rho \end{cases}$$

The position vector $\overrightarrow{OM}$ in polar coordinates is then written as:

$$\overrightarrow{OM} = \|\overrightarrow{OM}\|\,\vec{e}_\rho = \rho\,\vec{e}_\rho \tag{2.8}$$

2.2.2 Elementary displacement

The elementary displacement in polar coordinates is obtained by differentiating the position vector $\overrightarrow{OM}$:

$$d\overrightarrow{OM} = d(\rho\vec{e}_\rho) = d\rho\vec{e}_\rho + \rho d\vec{e}_\rho = d\rho\vec{e}_\rho + \rho d\varphi\vec{e}_\varphi \tag{2.9}$$

2.3 Cylindrical coordinate system

2.3.1 Definition

In cylindrical coordinates, we use a basis that we will denote $(\vec{e}_\rho, \vec{e}_\varphi, \vec{k})$. This basis is used in all problems where symmetry is based on revolution around an axis that is arbitrarily set as *the z- axis*. The basis is referenced to the coordinate system (*O, X, Y, Z*) by the angle φ formed by the vector $\vec{e}_\rho$ with the *X-axis*.

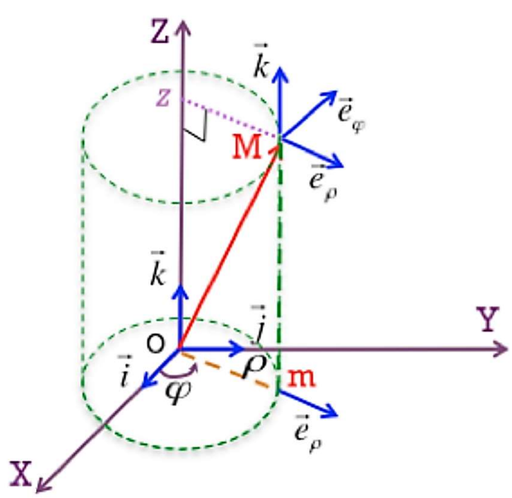


Figure 2.5 – Representation of the cylindrical coordinate system.

The projection *m* of point *M* onto the plane (*O, X, Y*) is marked in polar coordinates (ρ, φ). The projection of *M* onto the axis (*OZ*) gives the coordinate z, as shown in the figure (2.5). When point *M* describes space, the intervals of variation of its coordinates are:

$$0 \leq \rho \leq +\infty \quad , \quad 0 \leq \varphi < 2\pi \quad , \quad -\infty < z < +\infty \tag{2.10}$$

We can convert from cylindrical coordinates to Cartesian coordinates using the following relationships:

$$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \\ z = z \end{cases} \tag{2.11}$$

or conversely:

$$\begin{cases} \rho = \sqrt{x^2 + y^2} \\ \varphi = \arctan\dfrac{y}{x} \\ z = z \end{cases} \tag{2.12}$$

The basis of the cylindrical coordinate system is related to the basis of the Cartesian coordinates by the following relations:

$$\begin{cases} \vec{e}_\rho = \cos\varphi\,\vec{\imath} + \sin\varphi\,\vec{\jmath} \\ \vec{e}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \\ \vec{k} = \vec{k} \end{cases} \tag{2.13}$$

with:

$$\begin{cases} \dfrac{d\vec{e}_\rho}{d\varphi} = \vec{e}_\varphi \\ \dfrac{d\vec{e}_\varphi}{d\varphi} = -\vec{e}_\rho \\ d\vec{k} = \vec{0} \end{cases} \tag{2.14}$$

In this basis, the position vector $\overrightarrow{OM}$ is obtained using Chasles' relation:

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\vec{e}_\rho + z\vec{k} \tag{2.15}$$

2.3.2 Elementary displacement

Let M' be a point in space very close to point *M*, as illustrated in Figure 2.6.

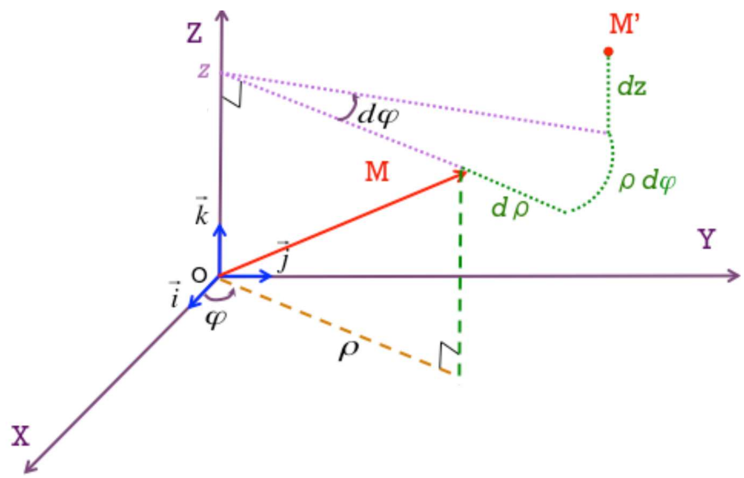


Figure 2.6 – Elementary displacement in cylindrical coordinates.

The elementary displacement vector $\overrightarrow{MM'}$ in cylindrical coordinates is given by:

$$\overrightarrow{MM'} = d\rho\, \vec{e}_\rho + \rho d\varphi\, \vec{e}_\varphi + dz\, \vec{k} \tag{2.16}$$

Indeed, the displacement from M to M' is achieved by performing a translation $d\rho$ along $\vec{e}_\rho$, followed by a rotation of an angle $d\varphi$, which results in a displacement of $\rho d\varphi$, then a translation dz along $\vec{k}$.

2.3.3 Elementary volume

The elementary volume dV generated from M by variations in the three parameters ρ, φ, and z as shown in Figure (2.7) is given by:

$$dV = \rho d\rho d\varphi dz \tag{2.17}$$

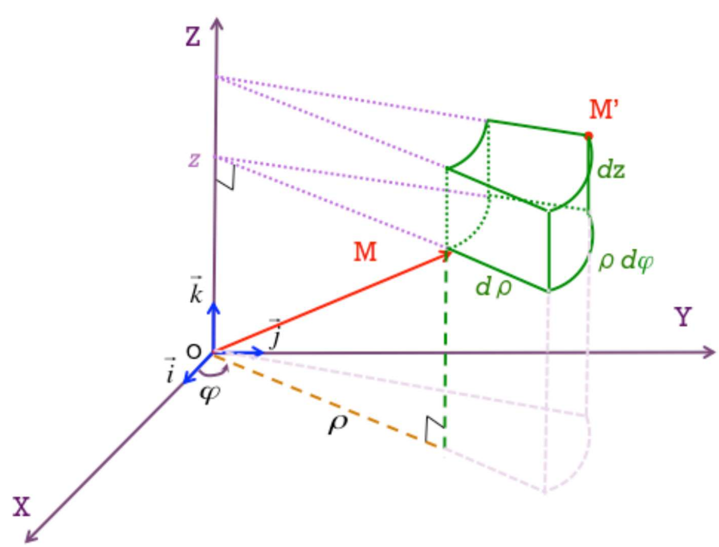


Figure 2.7 – Elementary volume in cylindrical coordinates.

2.4 Spherical coordinate system

2.4.1 Definition

This coordinate system, illustrated in Figure 2.8-a, is very useful in all problems involving spherical symmetry, a good example of which is locating a point on the surface of the Earth.

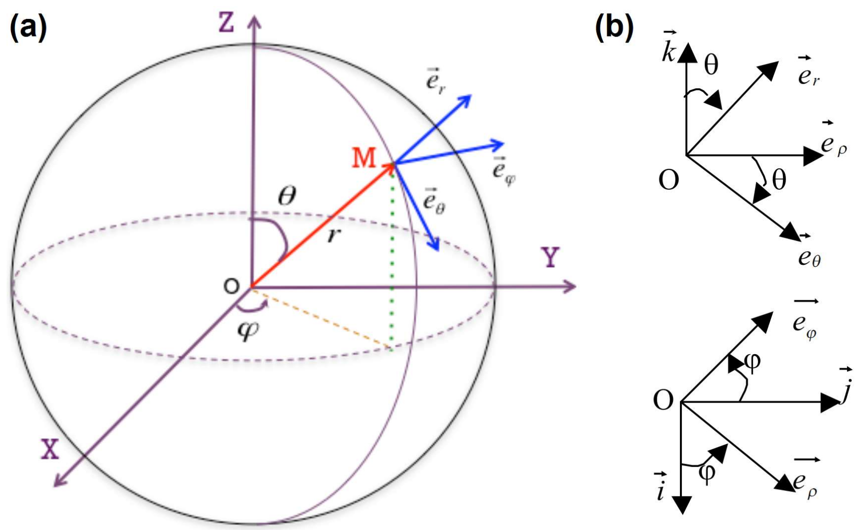


Figure 2.8 – (a) Illustration of the quantities used in the spherical coordinate system and (b) the associated basis ($\vec{e}_\theta$,$\vec{e}_\varphi$).

Point *M* is identified in this coordinate system by three position coordinates, r, φ, and θ, such that:

$$\begin{cases} r = \|\overrightarrow{OM}\| \\ \theta = \left(\widehat{\overrightarrow{OM},\vec{k}}\right) \\ \varphi = \left(\widehat{\overrightarrow{Om},\vec{\imath}}\right) \end{cases} \tag{2.18}$$

where *m* is the projection of *M* onto the horizontal plane.

When point *M* describes space, we have:

$$0 \leq r < +\infty \quad , \quad 0 \leq \theta \leq \pi \quad , \quad 0 \leq \varphi < 2\pi \tag{2.19}$$

We have a new rectangular coordinate system whose basis $(\vec{e}_r,\ \vec{e}_\theta,\ \vec{e}_\varphi)$ is orthonormal. Note that the vectors $\vec{e}_r$, $\vec{e}_\theta$ and $\vec{e}_\varphi$ depend on the position of point *M* and so they vary from one point to another in space. Only their norm remains constant since they are unitary.

As shown in Figure (2.8-b), the spherical coordinate basis is related to the Cartesian coordinate basis by the relations:

$$\begin{cases} \vec{e}_r = \sin\theta\,\vec{e}_\rho + \cos\theta\,\vec{k} = \sin\theta\cos\varphi\,\vec{\imath} + \sin\theta\cos\varphi\,\vec{\jmath} + \cos\theta\,\vec{k} \\ \vec{e}_\theta = \cos\theta\,\vec{e}_\rho - \sin\theta\,\vec{k} = \cos\theta\cos\varphi\,\vec{\imath} + \cos\theta\sin\varphi\,\vec{\jmath} - \sin\theta\,\vec{k} \\ \vec{e}_\varphi = -\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath} \end{cases} \tag{2.20}$$

If $\varphi = cste$, we have: $\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta$, $\frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$, $\frac{d\vec{e}_\varphi}{d\theta} = \vec{0}$
If $\theta = cste$, we have: $\frac{d\vec{e}_r}{d\varphi} = \sin\theta\,\vec{e}_\varphi$, $\frac{d\vec{e}_\theta}{d\varphi} = \cos\theta\,\vec{e}_\varphi$, $\frac{d\vec{e}_\varphi}{d\varphi} = -\vec{e}_\rho$

The spherical coordinates (r, θ, φ) of a point M are related to its Cartesian coordinates (x, y, z) by the following relationships:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \tag{2.21}$$

where:

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \arctan\frac{\sqrt{x^2 + y^2}}{z} \\ \varphi = \arctan\frac{y}{x} \end{cases} \tag{2.22}$$

In the basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$, the position vector is written as:

$$\overrightarrow{OM} = r\vec{e}_r \tag{2.23}$$

2.4.2 Elementary displacement

The elementary displacement of particle M in spherical coordinates is given by:

$$\begin{aligned} d\overrightarrow{OM} &= d(r\vec{e}_r) \\ &= dr\vec{e}_r + rd\vec{e}_r \\ &= dr\vec{e}_r + rd\theta\vec{e}_\theta + r\sin\theta d\varphi\vec{e}_\varphi \end{aligned} \tag{2.24}$$

since:

$$d\vec{e}_r = \frac{\partial\vec{e}_r}{\partial\theta}\,d\theta + \frac{\partial\vec{e}_r}{\partial\varphi}\,d\varphi = d\theta\,\vec{e}_\theta + \sin\theta\,d\varphi\vec{e}_\varphi \tag{2.25}$$

The same result can be obtained using a geometric approach. Indeed, as shown in Figure (2.9), a variation dr of r gives rise to a displacement $dr\vec{e}_r$, a variation $d\theta$ of θ gives rise to a displacement $rd\theta\vec{e}_\theta$, and a variation $d\varphi$ of φ gives rise to a displacement $r\sin\theta d\varphi\vec{e}_\varphi$.

2.4.3 Elementary volume

The volume element in spherical coordinates, as shown in Figure (2.9), is obtained by taking the product of the components of the elementary displacement:

$$dV = r^2 \sin\theta \, dr d\theta d\varphi \tag{2.26}$$

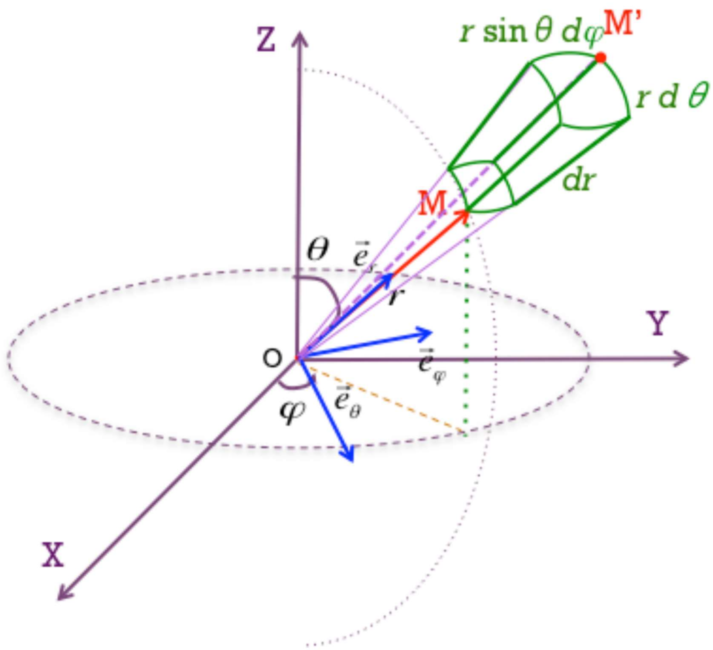


Figure 2.9 – Elementary volume in spherical coordinates.

2.5 Choice of coordinate system

The choice of coordinate system will depend on the type of movement of the moving point. In the case of rectilinear motion, it is clear that the Cartesian coordinate system is the most suitable. This is not the case for curvilinear motions, for which the polar or cylindrical coordinate system is most often used. On the other hand, spherical coordinate systems are rarely used because they lead to systems of equations that are difficult to solve analytically. However, their use is sometimes relevant, particularly when the properties of the moving object under study depend only on the distance to a point, for example, the study of the motion of a moving object relative to the center of the Earth.

Chapter 3

Kinematics of a material point without change of reference frame

The word Kinematics comes from the Greek word "kinêma," which means movement. Kinematics is the branch of mechanics that studies the motion of bodies, regardless of the causes that generate them. The purpose of the kinematics of a material point is to determine kinematic quantities such as **position vectors, velocity, acceleration**, and **the hourly equation of the trajectory** of this point relative to a reference frame chosen by the observer.

3.1 Material point

A material point is an "ideal object" located at its center of gravity, with no geometric dimensions, whose rotational movement around itself or spatial extension is negligible. It is modeled by a geometric point, generally denoted by *M*.

3.2 Reference frame or reference system

In mechanics, the description of the position or movement of an object is necessarily linked to a reference frame. A reference frame is a solid reference consisting of a set of points that are all fixed relative to each other.

A reference frame can be defined by one of its spatial coordinates equipped with an origin, three axes, and a timeline: R (O, X, Y, Z, t).

3.3 Trajectory

The trajectory of a moving point *M* in a given coordinate system is the curve formed by the set of successive positions of point *M* in that coordinate system. The trajectory of a moving point depends on the chosen reference frame. Indeed, let *M* be a moving point and *O* a fixed origin. At each instant *t,* the position of *M* is given by the vector $\overrightarrow{OM}(t)$. The set of positions of point *M* when *t* varies continuously forms a curve that represents the trajectory of the moving object.

3.4 Velocity vector of a material point

Since the trajectory of a moving point depends on the chosen reference frame, the characteristics of the motion must change from one reference frame to another. One of these characteristics is the velocity vector of the moving point. For this reason, we use the notation $\vec{V}\ (M/R)$ to

indicate that this is the velocity of point *M* relative to reference frame *R*. We will use the same notation for the two types of velocity that we will discuss below, average velocity and instantaneous velocity.

3.4.1 Average velocity

The average velocity represents the distance traveled by a moving object M during the travel time. Let us consider a point M occupying position $M = M(t)$ at time t and position $M' = M(t')$ at time t'(with $t' > t$) on the oriented trajectory (C), as shown in Figure 3.1.

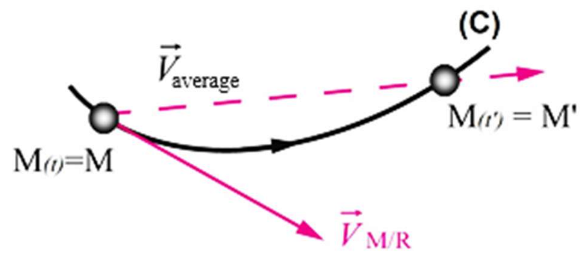


Figure 3.1 – Variation of position over time: average velocity.

So, the average velocity of point M between times t and t' is:

$$\vec{V}_{average} = \frac{\overrightarrow{MM'}}{t' - t} = \frac{\overrightarrow{OM}(t') - \overrightarrow{OM}(t)}{t' - t} \tag{3.1}$$

where point O is the origin of the reference space from which the average velocity of point M is determined.

3.4.2 Instantaneous velocity

The instantaneous velocity is the limit of $\vec{V}_{average}$ when t' approaches t. Let $t' = t + \delta t$ (with δt *being* an infinitesimal variation of t), the instantaneous velocity of point M is then written as:

$$\vec{V}(t) = \lim_{t' \to t} \frac{\overrightarrow{MM'}}{t' - t} = \lim_{\delta t \to 0} \frac{\overrightarrow{OM}(t + \delta t) - \overrightarrow{OM}(t)}{\delta t} = \frac{d\overrightarrow{OM}(t)}{dt} \tag{3.2}$$

The instantaneous velocity of a point M is therefore the derivative with respect to time of the position vector $\overrightarrow{OM}(t)$. It is a vector that is always tangent to the trajectory of point M and directed in the direction of the movement of point M on (C). Subsequently, the instantaneous velocity of point M, at time t, relative to a spatial reference frame (R) will be denoted $\vec{V}_{M/R}(t)$ or more simply $\vec{V}_{M/R}$, such that:

$$\vec{V}_{M/R} = \left.\frac{d\overrightarrow{OM}(t)}{dt}\right|_R \tag{3.3}$$

3.4.3 Velocity vector in different coordinate systems

Velocity in Cartesian coordinates

When the reference frame in which the motion is studied is Cartesian, the position of point M is written as:

$$\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k} \tag{3.4}$$

The basis vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ of the Cartesian coordinates are constant, their derivatives with respect to time are zero:

$$\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = \vec{0} \quad (3.5)$$

and the derivative of the position leads to:

$$\vec{V}_{M/R} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\left(x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}\right)}{dt} = \frac{dx}{dt}\,\vec{\imath} + \frac{dy}{dt}\,\vec{\jmath} + \frac{dz}{dt}\,\vec{k} \quad (3.6)$$

The previous notation can be condensed by using variables with a dot above them to describe the time derivative. The velocity is then written as follows:

$$\vec{V}_{M/R} = \dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k} \quad (3.7)$$

Velocity in polar coordinates

When the point moves in a plane, it is possible to choose the polar coordinate system. The associated basis $(\vec{e}_\rho, \vec{e}_\varphi)$ is then a moving basis: the vectors rotate in the plane (*O, X, Y*) and are therefore functions of time.

By applying the definition of velocity, it is possible to express the velocity vector of the point *M* in the mobile base, i.e.:

$$\vec{V}_{M/R} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\left(\rho\vec{e}_\rho\right)}{dt} = \frac{d\rho}{dt}\,\vec{e}_\rho + \rho\frac{d\vec{e}_\rho}{dt} \quad (3.8)$$

Lorsque le point *M* est en mouvement, l'angle polaire $\varphi = \varphi(t)$ est une fonction du temps. Le vecteur unitaire $\vec{e}_\rho$ tourne alors au cours du temps et est donc fonction du temps par l'intermédiaire de l'angle. L'expression de la dérivée par rapport au temps du vecteur unitaire tournant $\vec{e}_\rho$ s'écrit :

$$\frac{d\vec{e}_\rho}{dt} = \frac{d\vec{e}_\rho}{d\varphi}\frac{d\varphi}{dt} = \dot{\varphi}\vec{e}_\varphi \quad (3.9)$$

which means that in polar coordinates:

$$\vec{V}_{M/R} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi = V_\rho\,\vec{e}_\rho + V_\varphi\,\vec{e}_\varphi \quad (3.10)$$

where $V_\rho = \dot{\rho}$ and $V_\varphi = \rho\dot{\varphi}$ are respectively the radial and orthoradial components of the velocity vector velocity vector in the polar basis.

Velocity in cylindrical coordinates

Cylindrical coordinates correspond to polar coordinates in the plane (*O, X, Y*) to which a *z* coordinate is added along an axis perpendicular to the plane. The associated basis is therefore composed of the rotating basis $(\vec{e}_\rho, \vec{e}_\varphi)$ and the vector $\vec{k}$, which is a fixed vector in the reference frame under study (its derivative is zero: $\frac{d\vec{k}}{dt} = 0$). To obtain the expression of the velocity vector in cylindrical coordinates, simply add the component following $\vec{k}$:

$$\vec{V}_{M/R} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\left(\rho\,\vec{e}_\rho + z\vec{k}\right)}{dt} = \frac{d\left(\rho\,\vec{e}_\rho\right)}{dt} + \frac{d\left(z\vec{k}\right)}{dt} = \frac{d\rho}{dt}\,\vec{e}_\rho + \rho\frac{d\vec{e}_\rho}{dt} + \frac{dz}{dt}\vec{k} \tag{3.11}$$

where:

$$\vec{V}_{M/R} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\,\vec{k} \tag{3.12}$$

Velocity in spherical coordinates

The instantaneous velocity vector of particle *M* in the *R* coordinate system in spherical coordinates is written as:

$$\vec{V}_{M/R} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r\vec{e}_r)}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} \tag{3.13}$$

The position vector in spherical coordinates therefore depends on the vector $\vec{e}_r$. The latter depends on the angles θ and φ, so its derivative with respect to time is given by:

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{d\vec{e}_r}{d\varphi}\frac{d\varphi}{dt} \tag{3.14}$$

Using the expressions of the vectors$\left(\vec{e}_r,\ \vec{e}_\theta,\ \vec{e}_\varphi\right)$ as a function of the vectors $\left(\vec{\imath}\,,\vec{\jmath}\,,\vec{k}\right)$ given in the second chapter, we show that:

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad et \qquad \frac{d\vec{e}_r}{d\varphi} = \sin\theta\,\vec{e}_\varphi \tag{3.15}$$

Thus:

$$\frac{d\vec{e}_r}{dt} = \frac{d\theta}{dt}\vec{e}_\theta + \sin\theta\,\frac{d\varphi}{dt}\vec{e}_\varphi \tag{3.16}$$

The velocity vector is then:

$$\vec{V}_{M/R} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \mathrm{r}\sin\theta\frac{d\varphi}{dt}\vec{e}_\varphi \tag{3.17}$$

or:

$$\vec{V}_{M/R} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \mathrm{r}\sin\theta\dot{\varphi}\vec{e}_\varphi \tag{3.18}$$

3.5 Acceleration vector of a material point

3.5.1 Definition

The acceleration of a material point *M* relative to a reference frame *R* is defined as the derivative of the velocity vector with respect to time, i.e.:

$$\vec{\gamma}_{M/R} = \frac{d\vec{V}_{M/R}}{dt} = \frac{d}{dt}\left(\frac{d\overrightarrow{OM}}{dt}\right) = \frac{d^2\overrightarrow{OM}}{dt^2} \tag{3.19}$$

Acceleration is also the second derivative of position with respect to time.

3.5.2 Acceleration vector in different coordinate systems

Acceleration in Cartesian coordinates

Consider an orthonormal Cartesian basis $(\vec{\imath},\vec{\jmath},\vec{k})$ of the reference frame R used to define the position of point M . The acceleration of point M in this basis is written as follows, since the basis vectors $\vec{\imath}$, $\vec{\jmath}$ and $\vec{k}$ are constant:

$$\vec{\gamma}_{M/R} = \frac{d\vec{V}_{M/R}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k} \tag{3.20}$$

with the following notation:

$$\ddot{x} = \frac{d^2x}{dt^2}, \qquad \ddot{y} = \frac{d^2y}{dt^2}, \qquad \ddot{z} = \frac{d^2z}{dt^2} \tag{3.21}$$

where the two dots above a variable denote the second derivative of the variable with respect to time.

Acceleration in polar coordinates

If we use the polar basis $(\vec{e}_\rho, \vec{e}_\varphi)$ as the reference basis, which is a basis that rotates with the position of point M in the plane (O, X, Y), we have shown that the velocity in this basis can be written as: $\vec{V}_{M/R} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi$.

The acceleration of point M relative to reference frame R is expressed in this basis by:

$$\begin{aligned}
\vec{\gamma}_{M/R} &= \frac{d\vec{V}_{M/R}}{dt} \\
&= \frac{d\left(\dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\,\vec{e}_\varphi\right)}{dt} \\
&= \frac{d\left(\dot{\rho}\,\vec{e}_\rho\right)}{dt} + \frac{d\left(\rho\dot{\varphi}\,\vec{e}_\varphi\right)}{dt} \\
&= \frac{d(\dot{\rho})}{dt}\vec{e}_\rho + \dot{\rho}\frac{d\left(\vec{e}_\rho\right)}{dt} + \frac{d(\rho)}{dt}\dot{\varphi}\vec{e}_\varphi + \rho\frac{d(\dot{\varphi})}{dt}\vec{e}_\varphi + \rho\dot{\varphi}\frac{d\left(\vec{e}_\varphi\right)}{dt} \\
&= \ddot{\rho}\vec{e}_\rho + \dot{\rho}\frac{d\left(\vec{e}_\rho\right)}{dt} + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi + \rho\dot{\varphi}\frac{d\left(\vec{e}_\varphi\right)}{dt}
\end{aligned} \tag{3.22}$$

We obtained the expression for the derivative with respect to time of the vector $\vec{e}_\rho$:

$$\frac{d\vec{e}_\rho}{dt} = \frac{d\vec{e}_\rho}{d\varphi}\frac{d\varphi}{dt} = \dot{\varphi}\vec{e}_\varphi \tag{3.23}$$

Similarly, we obtain the derivative of the vector $\vec{e}_\varphi$:

$$\frac{d\vec{e}_\varphi}{dt} = \frac{d\vec{e}_\varphi}{d\varphi}\frac{d\varphi}{dt} = \frac{d(-\sin\varphi\,\vec{\imath} + \cos\varphi\,\vec{\jmath})}{d\varphi}\frac{d\varphi}{dt} = (-\cos\varphi\,\vec{\imath} - \sin\varphi\,\vec{\jmath})\frac{d\varphi}{dt} = -\dot{\varphi}\vec{e}_\rho \tag{3.24}$$

Substituting in the expression for acceleration above, we obtain:

$$\begin{aligned}\vec{\gamma}_{M/R} &= \ddot{\rho}\vec{e}_\rho + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \dot{\rho}\dot{\varphi}\vec{e}_\varphi + \rho\ddot{\varphi}\vec{e}_\varphi - \rho\dot{\varphi}^2\vec{e}_\rho \\ &= (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi \\ &= \gamma_\rho\vec{e}_\rho + \gamma_\varphi\vec{e}_\varphi\end{aligned} \tag{3.25}$$

The first term ($\gamma_\rho = \ddot{\rho} - \rho\dot{\varphi}^2$) corresponds to the radial component of acceleration, the second ($\gamma_\varphi = 2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi}$) to orthoradial acceleration.

Acceleration in cylindrical coordinates

In cylindrical coordinates, simply add the third component along the axis (*OZ*):

$$\vec{V}_{M/R} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\,\vec{k} \tag{3.26}$$

The expression for the acceleration vector is obtained by adding the $\ddot{z}$ component along $\vec{k}$:

$$\vec{\gamma}_{M/R} = \frac{d\vec{V}_{M/R}}{dt} = \frac{d\left(\dot{\rho}\,\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi + \dot{z}\,\vec{k}\right)}{dt} = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi + \dot{z}\,\vec{k} \tag{3.27}$$

Acceleration in spherical coordinates

From the expression of the velocity vector in spherical coordinates and the definition of the acceleration vector, we obtain:

$$\begin{aligned}\vec{\gamma}_{M/R} &= \frac{d\vec{V}_{M/R}}{dt} \\ &= \frac{d\left(\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\,\dot{\varphi}\vec{e}_\varphi\right)}{dt} \\ &= \frac{d(\dot{r}\vec{e}_r)}{dt} + \frac{d\left(r\dot{\theta}\vec{e}_\theta\right)}{dt} + \frac{d\left(r\sin\theta\,\dot{\varphi}\vec{e}_\varphi\right)}{dt}\end{aligned}$$

$$= \frac{d(\dot{r})}{dt}\vec{e}_r + \dot{r}\frac{d(\vec{e}_r)}{dt} + \frac{d(r)}{dt}\dot{\theta}\vec{e}_\theta + r\frac{d(\dot{\theta})}{dt}\vec{e}_\theta + r\dot{\theta}\frac{d(\vec{e}_\theta)}{dt} + \frac{d(r)}{dt}\sin\theta\,\dot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\frac{d(\sin\theta)}{dt}\vec{e}_\varphi + r\sin\theta\frac{d(\dot{\varphi})}{dt}\vec{e}_\varphi + r\sin\theta\,\dot{\varphi}\frac{d\left(\vec{e}_\varphi\right)}{dt}$$

$$= \ddot{r}\vec{e}_r + \dot{r}\frac{d(\vec{e}_r)}{dt} + \dot{r}\dot{\theta}\vec{e}_\theta + r\ddot{\theta}\vec{e}_\theta + r\dot{\theta}\frac{d(\vec{e}_\theta)}{dt} + \dot{r}\sin\theta\,\dot{\varphi}\vec{e}_\varphi + r\dot{\varphi}\cos\theta\,\vec{e}_\varphi + r\sin\theta\,\ddot{\varphi}\vec{e}_\varphi + r\sin\theta\,\dot{\varphi}\frac{d\left(\vec{e}_\varphi\right)}{dt}$$

(3.28)

To derive the basis vectors $\left(\vec{e}_r,\ \vec{e}_\theta,\ \vec{e}_\varphi\right)$, we use their expressions as functions of the basis vectors $\left(\vec{\imath}\,,\vec{\jmath}\,,\vec{k}\right)$. We then obtain:

$$\begin{aligned}&\frac{\partial\vec{e}_r}{\partial\theta} = \vec{e}_\theta \quad , \quad \frac{\partial\vec{e}_r}{\partial\varphi} = \sin\theta\,\vec{e}_\varphi \\ &\frac{\partial\vec{e}_\theta}{\partial\theta} = -\vec{e}_r \quad , \quad \frac{\partial\vec{e}_\theta}{\partial\varphi} = \cos\theta\,\vec{e}_\varphi \\ &\frac{\partial\vec{e}_\varphi}{\partial\varphi} = -\vec{e}_\rho = -(\sin\theta\,\vec{e}_r + \cos\theta\,\vec{e}_\theta)\end{aligned}$$

The time derivatives of the basis vectors $\left(\vec{e}_r,\ \vec{e}_\theta,\ \vec{e}_\varphi\right)$ are then given by:

$$\frac{d\vec{e}_r}{dt} = \frac{\partial \vec{e}_r}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \vec{e}_r}{\partial \varphi}\frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{d\vec{e}_r}{dt} = \dot{\theta}\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi$$

$$\frac{d\vec{e}_\theta}{dt} = \frac{\partial \vec{e}_\theta}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \vec{e}_\theta}{\partial \varphi}\frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{d\vec{e}_\theta}{dt} = -\dot{\theta}\vec{e}_r + \dot{\varphi}\cos\theta\,\vec{e}_\varphi$$

$$\frac{d\vec{e}_\varphi}{dt} = \frac{\partial \vec{e}_\varphi}{\partial \varphi}\frac{d\varphi}{dt} \quad \Rightarrow \quad \frac{d\vec{e}_\varphi}{dt} = -\dot{\varphi}\sin\theta\,\vec{e}_r - \dot{\varphi}\cos\theta\,\vec{e}_\theta$$

The final expression of the acceleration vector in spherical coordinates is obtained by replacing the time derivatives of the basis vectors $(\vec{e}_r,\ \vec{e}_\theta,\ \vec{e}_\varphi)$ with their respective expressions detailed:

$$\vec{\gamma}_{M/R} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\vec{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\vec{e}_\theta + (2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta)\vec{e}_\varphi \quad (3.29)$$

3.6 Examples of movements

3.6.1 Rectilinear motions

Uniform rectilinear motion

The motion of a material point is said to be uniform rectilinear if the material point moves at a constant velocity vector.

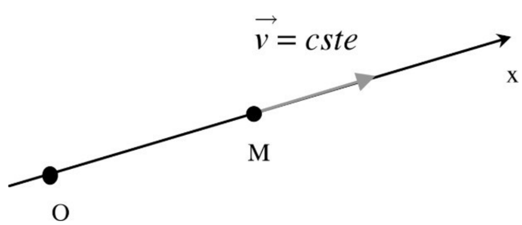


Figure 3.2 – Uniform rectilinear motion; point *M* moves along a straight line at a constant velocity.

$$\text{Uniform rectilinear motion} \quad \Leftrightarrow \quad \vec{V} = \overrightarrow{cste}.$$

Since the velocity vector is constant, the motion is rectilinear because the velocity is tangent to the trajectory. The line on which the point moves is assimilated to the *x-axis*. The differential equation of motion is then written as:

$$\vec{V} = \dot{x}\,\vec{\imath} = C\,\vec{\imath} \Rightarrow \dot{x} = C \quad (3.30)$$

which leads to the following time equation:

$$x = Ct + x_0 \quad (3.31)$$

Uniformly varied motion

A motion is said to be uniformly varied if the acceleration vector is constant and the trajectory is straight.

Uniformly varied rectilinear motion ⇔ $\vec{\gamma} = \overrightarrow{cste}$ and straight trajectory.

If the motion is rectilinear, it is convenient to set the axis of motion as the axis of x. We therefore have:

$$\overrightarrow{OM} = x\,\vec{\imath} \Longrightarrow \vec{V} = \dot{x}\,\vec{\imath} \Longrightarrow \vec{\gamma} = \ddot{x}\,\vec{\imath} \tag{3.32}$$

and

$$\vec{\gamma} = \ddot{x}\,\vec{\imath} = C\,\vec{\imath} \tag{3.33}$$

By integrating this equation, we obtain the velocity of point M:

$$V = \dot{x} = Ct + B \tag{3.34}$$

which, through further integration, leads to the hourly equation of motion:

$$x = \frac{1}{2}Ct^2 + Bt + D \tag{3.35}$$

The constants B and D that appeared in the two successive integrations are determined by the initial conditions of the motion of point M. Thus, if point M has zero velocity and is at $x = x_0$ at $t = 0$, the constants B and D become $B = 0$ and $D = x_0$, and the equation of motion is then written as:

$$x = \frac{1}{2}Ct^2 + x_0 \tag{3.36}$$

Notes: The motion is uniformly accelerated if the norm of the velocity vector is an increasing function of t, i.e., V^2 is an increasing function. The derivative of V^2 must therefore be positive.
The condition will be:

$$\frac{dV^2}{dt} > 0 \Longrightarrow 2V.\frac{dV}{dt} > 0 \tag{3.37}$$

Studying the sign of the product of velocity and acceleration will determine whether the motion is *accelerated* ($\dot{x}.\ddot{x} > 0$) or *decelerated* ($\dot{x}.\ddot{x} < 0$).
Having a constant acceleration vector is not enough to say that the motion is rectilinear. The velocity vector must also have the same direction as the acceleration vector. Otherwise, the motion will be parabolic.

Sinusoidal rectilinear motion

The motion of a point M is said to be sinusoidal rectilinear if, occurring on an axis (OX), the abscissa x of point M is written as:

$$x = X_m \cos(\omega t + \varphi) \tag{3.38}$$

The term $\omega t + \varphi$ is called the phase at time t, with ω being the phase at the origin of the dates ($t = 0$). The term X_m corresponds to the amplitude of the movement, with x varying sinusoidally from $-X_m$ to X_m as shown in Figure (3.3).

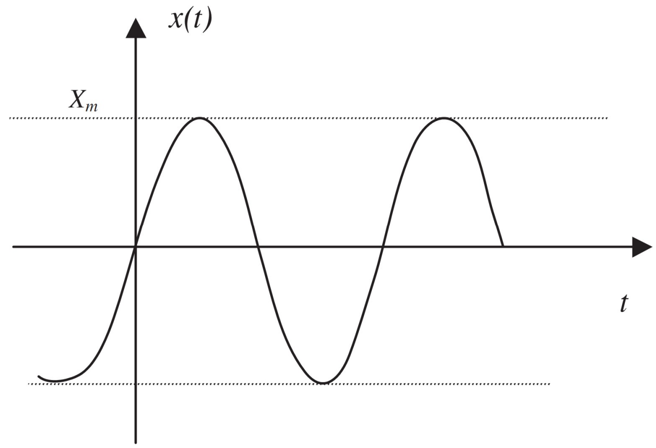


Figure 3.3 – Representation of sinusoidal motion over time.

The velocity is expressed as:

$$V = \dot{x} = -\omega X_m \sin(\omega t + \varphi) \tag{3.39}$$

and acceleration is expressed as:

$$\gamma = \ddot{x} = -\omega^2 X_m \cos(\omega t + \varphi) = -\omega^2 x \tag{3.40}$$

The differential equation of motion is therefore:

$$\ddot{x} + \omega^2 x = 0 \tag{3.41}$$

This equation corresponds to the second-order differential equation of a harmonic oscillator.
Note: The solution to this differential equation can be written in different ways, all of which are equivalent. We have:

$$x = X_m \cos(\omega t + \varphi) = X_m \sin(\omega t + \varphi') = A \sin \omega t + B \cos \omega t \tag{3.42}$$

Using the usual trigonometric relations, we obtain very simply:

$$\varphi' = \varphi + \pi/2 \quad ; \quad A = -X_m \sin \varphi \quad ; \quad B = X_m \cos \varphi \tag{3.43}$$

3.6.2 Uniform circular motion

The motion of a point is said to be uniform circular if:
- the point moves along a circle;
- its angular velocity of rotation is constant

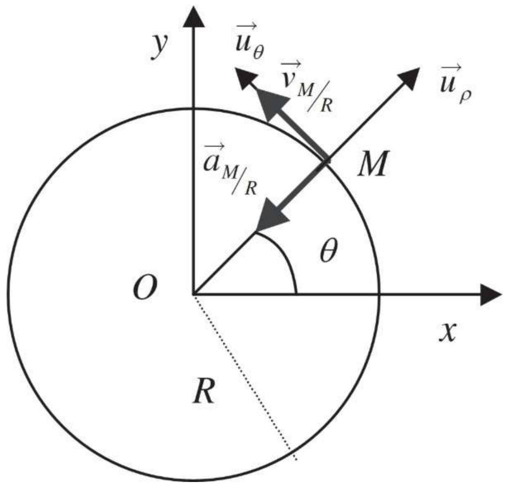


Figure 3.4 – Uniform circular movement.

The differential equation of motion is given by:

$$\frac{d\theta}{dt} = \omega = cste \tag{3.44}$$

which leads by integration to:

$$\theta = \omega t + \theta_0 \tag{3.45}$$

The kinematic characteristics of uniform circular motion can be deduced from the diagram in Figure (3.4) and are given by:

$$\begin{aligned}
\overrightarrow{OM}(t) &= \rho\vec{u}_\rho(t) = \rho\cos\theta\,\vec{u}_x + \rho\sin\theta\,\vec{u}_y,\\
\vec{V}(t) &= \frac{d\left(\rho\vec{u}_\rho(t)\right)}{dt} = \rho\dot{\theta}\vec{u}_\theta(t),\\
\vec{\gamma}(t) &= \frac{d\vec{V}(t)}{dt} = -\rho\dot{\theta}^2\vec{u}_\rho(t)
\end{aligned}$$

We therefore note that uniform circular motion is accelerated motion with centripetal acceleration. Noting that $\vec{u}_\theta = \vec{u}_z \wedge \vec{u}_\rho$,we can give an expression for the velocity vector that is independent of the chosen basis. Indeed, we obtain:

$$\vec{V}(t) = \rho\dot{\theta}\vec{u}_\theta(t) = \rho\dot{\theta}\vec{u}_z \wedge \vec{u}_\rho(t) = \dot{\theta}\vec{u}_z \wedge \rho\vec{u}_\rho(t) = \vec{\omega} \wedge \overrightarrow{OM}(t) \tag{3.46}$$

In this expression, $\vec{\omega}$ is the angular velocity vector. This relationship is valid for any circular motion. Similarly, we obtain the following for the acceleration vector:

$$\vec{\gamma}(t) = \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{OM}(t)\right) = \vec{\omega} \wedge \vec{V}(t) \tag{3.47}$$

This result can be obtained directly by deriving the velocity vector expressed as a vector product:

$$\vec{\gamma}(t) = \frac{d\vec{V}(t)}{dt} = \frac{d\left(\vec{\omega} \wedge \overrightarrow{OM}(t)\right)}{dt} = \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{OM}(t) + \vec{\omega} \wedge \frac{d\overrightarrow{OM}(t)}{dt} \tag{3.48}$$

If the motion is uniform circular, the angular velocity vector $\vec{\omega}$ is a constant vector. Since its derivative is zero, we find the expression for the acceleration vector.

3.6.3 Central acceleration motion

A centrally accelerated motion is a motion in which the acceleration of particle M, $\vec{\gamma}(M/R)$, is parallel to the position vector $\overrightarrow{OM}$ at all times t. It follows that:

$$\overrightarrow{OM} \wedge \vec{\gamma}(M/R) = \vec{0} \tag{3.49}$$

Furthermore:

$$\overrightarrow{OM} \wedge \vec{\gamma}(M/R) = \frac{d\left[\overrightarrow{OM} \wedge \vec{V}(M/R)\right]}{dt} = \vec{0} \tag{3.50}$$

because

$$\frac{d\left[\overrightarrow{OM} \wedge \vec{V}(M/R)\right]}{dt} = \frac{d\overrightarrow{OM}}{dt} \wedge \vec{V}(M/R) + \overrightarrow{OM} \wedge \frac{d\vec{V}(M/R)}{dt} \tag{3.51}$$

$$= \underbrace{\vec{V}(M/R) \wedge \vec{V}(M/R)}_{=\vec{0}} + \underbrace{\overrightarrow{OM} \wedge \vec{\gamma}(M/R)}_{=\vec{0}} \tag{3.52}$$

$$\Longrightarrow \frac{d\left[\overrightarrow{OM} \wedge \vec{V}(M/R)\right]}{dt} = \vec{0} \tag{3.53}$$

from where

$$\overrightarrow{OM} \wedge \vec{V}(M/R) = \overrightarrow{cste} = \vec{C} \tag{3.54}$$

$\vec{C}$ is a constant vector in magnitude, direction, and orientation. $\vec{C}$ is therefore perpendicular in the plane formed by $\overrightarrow{OM}$ and $\vec{V}(M/R)$. The position vector $\overrightarrow{OM}$ and the velocity vector $\vec{V}(M/R)$ therefore belong to the same plane regardless of the instant t considered.

Consequently, **any motion with central acceleration is a planar motion**. To study the motion of point M, it is therefore preferable to use its polar coordinates.

We recall that in the general case of a planar motion, the position, velocity, and acceleration vectors are written, respectively, as follows:

$$\begin{gathered} \overrightarrow{OM} = \rho\vec{e}_\rho, \\ \vec{V}(M/R) = \dot{\rho}\vec{e}_\rho + \rho\dot{\varphi}\vec{e}_\varphi, \\ \vec{\gamma}(M/R) = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho + (2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi})\vec{e}_\varphi \end{gathered} \tag{3.55}$$

Since the acceleration of point M is central (parallel to the position vector), it must be written in this case as:

$$\vec{\gamma}(M/R) = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho \tag{3.56}$$

and therefore its orthoradial component is zero:

$$2\dot{\rho}\dot{\varphi} + \rho\ddot{\varphi} = 0 \tag{3.57}$$

which can be written as:

$$\frac{1}{\rho}\frac{d(\rho^2\dot{\varphi})}{dt} = 0 \Longrightarrow \rho^2\dot{\varphi} = cste = C \tag{3.58}$$

Finally, $\rho^2\dot{\varphi} = C = \left|\overrightarrow{OM} \wedge \vec{V}(M/R)\right|$, called the area constant.

3.7 Law of areas

Let us calculate the area swept, per unit of time, by the radius vector $\overrightarrow{OM} = \rho\vec{e}_\rho$.

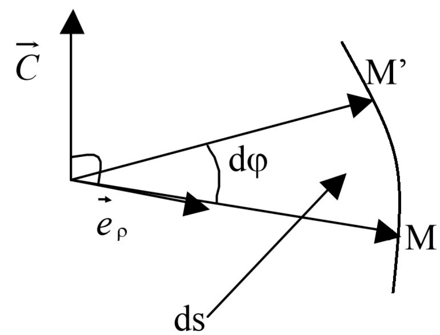


Figure 3.5 – The elementary area *dS* swept by the radius *OM* during the time *dt*.

$$\vec{C} = \overrightarrow{OM} \wedge \vec{V}(M/R) \tag{3.59}$$

$dS = \frac{1}{2}\left|\overrightarrow{OM} \wedge \overrightarrow{MM'}\right|$, M' is very close to *M*. Therefore:

$$dS = \frac{1}{2}\left|\rho\vec{e}_\rho \wedge \left(d\rho\vec{e}_\rho + \rho d\varphi\vec{e}_\varphi\right)\right| = \frac{1}{2}\rho^2 d\varphi \tag{3.60}$$

and

$$\frac{dS}{dt} = \frac{C}{2} \quad \text{from where} \quad \mathrm{dS} = \frac{C}{2}dt \quad \text{and} \quad \int_0^S dS = \frac{C}{2}\int_0^t dt \tag{3.61}$$

Therefore, $S = \frac{C}{2}t + S_0$ where $\frac{C}{2}$ is the areal velocity (cm^2/s). This result is known as Kepler's second law

3.8 Binet's formulas

3.8.1 Case of velocity

In the case of central acceleration motion, the square of the velocity vector modulus is:

$$V^2 = \dot{\rho}^2 + \rho^2\dot{\varphi}^2 \tag{3.62}$$

with $\dot{\rho} = \frac{d\rho}{dt} = \frac{d\rho}{d\varphi}\frac{d\varphi}{dt}$. We set $u = \frac{1}{\rho}$, therefore $du = -\frac{d\rho}{\rho^2}$ and $\frac{du}{d\varphi} = -\frac{1}{\rho^2}\frac{d\rho}{d\varphi}$, which gives:

$$\frac{d\rho}{d\varphi} = -\frac{1}{u^2}\frac{du}{d\varphi} \tag{3.63}$$

On the other hand, $C = \rho^2\dot{\varphi}$ can be written as $\dot{\varphi} = Cu^2$. Hence

$$V^2 = \left[-\left(\frac{1}{u^2}\frac{du}{d\varphi}\right)\right]^2 .C^2u^4 + \frac{1}{u^2}.C^2u^4 \tag{3.64}$$

Binet's first formula is written as:

$$V^2 = C^2\left[\left(\frac{du}{dt}\right)^2 + u^2\right] \tag{3.65}$$

This formula allows us to determine the polar equation $\rho = \rho\ (\varphi)$ or $u = u\ (\varphi)$ knowing the velocity of point M and vice versa.

3.8.2 Case of acceleration

Binet's second formula allows us to determine the acceleration of the particle under study if we know the polar equation, and vice versa.
Since the motion of point M is centrally accelerated, we have $\vec{\gamma}(M/R) = (\ddot{\rho} - \rho\dot{\varphi}^2)\vec{e}_\rho$, whose algebraic value is $\vec{\gamma} = \ddot{\rho} - \rho\dot{\varphi}^2$, with:

$$\ddot{\rho} = \frac{d\dot{\rho}}{d\varphi}\frac{d\varphi}{dt} = \frac{du}{d\varphi}\left(-C\frac{du}{d\varphi}\right).Cu^2 = -C^2u^2\frac{d^2u}{d\varphi^2} \tag{3.66}$$

and

$$\rho\dot{\varphi}^2 = \frac{1}{u}C^2u^4 = C^2u^3 \tag{3.67}$$

The second Binet’s formula can then be written as:

$$\gamma = -C^2u^2\frac{d^2u}{d\varphi^2} - C^2u^3 = -C^2u^2\left[\frac{d^2u}{d\varphi^2} + u\right] \tag{3.68}$$

Chapter 4

Kinematics with change of reference frame

Consider the motion of a particle M relative to a fixed reference frame R_1, called the absolute reference frame. It is sometimes useful to introduce a second reference frame R_2, called the relative reference frame, relative to which the motion of M is easier to study. Let

- $R_1(O_1, x_1, y_1, z_1)$ an absolute reference frame (fixed reference frame).
- $R_2(O_2, x_2, y_2, z_2)$ is a relative reference frame (a reference frame that moves relative to R_1).

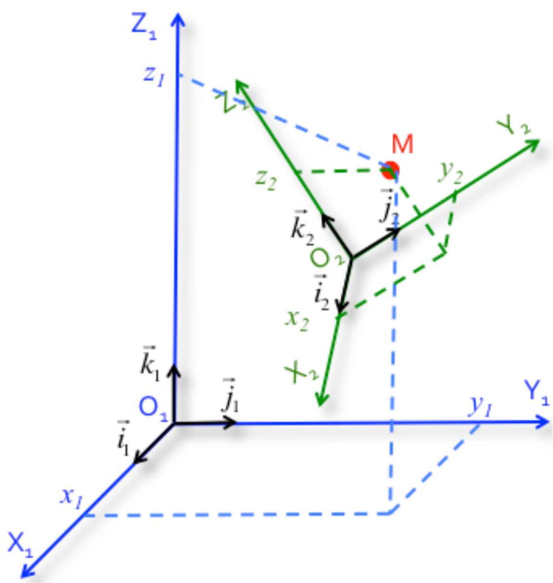


Figure 4.1 – Absolute coordinate system and relative coordinate system.

R_2 can be animated with a translational and/or rotational movement relative to R_1. The rotation of R_2 relative to R_1 occurs at an angular velocity $\vec{\omega}(R_2/R_1)$ such that in the R_1 coordinate system,

$$\left.\frac{d\vec{i}_2}{dt}\right|_{R_1} = \vec{\omega}\,(R_2/R_1) \wedge \vec{i}_2, \qquad \left.\frac{d\vec{j}_2}{dt}\right|_{R_1} = \vec{\omega}\,(R_2/R_1) \wedge \vec{j}_2, \qquad \left.\frac{d\vec{k}_2}{dt}\right|_{R_1} = \vec{\omega}\,(R_2/R_1) \wedge \vec{k}_2 \tag{4.1}$$

In the reference frame R_2:

$$\left.\frac{d\vec{i}_2}{dt}\right|_{R_2} = \left.\frac{d\vec{j}_2}{dt}\right|_{R_2} = \left.\frac{d\vec{k}_2}{dt}\right|_{R_2} = \vec{0} \tag{4.2}$$

4.1 Derivation in a moving reference frame

Let $\vec{A}$ be any vector. In the coordinate system R_1, this vector is written as:

$$\vec{A} = x_1\vec{\imath}_1 + y_1\vec{\jmath}_1 + z_1\vec{k}_1 \tag{4.3}$$

In the coordinate system R_2 ,the vector $\vec{A}$ is written as

$$\vec{A} = x_2\vec{\imath}_2 + y_2\vec{\jmath}_2 + z_2\vec{k}_2 \tag{4.4}$$

where:

$$\left.\frac{d\vec{A}}{dt}\right|_{R_1} = \dot{x}_1\vec{\imath}_1 + \dot{y}_1\vec{\jmath}_1 + \dot{z}_1\vec{k}_1 \tag{4.5}$$

$$\left.\frac{d\vec{A}}{dt}\right|_{R_2} = \dot{x}_2\vec{\imath}_2 + \dot{y}_2\vec{\jmath}_2 + \dot{z}_2\vec{k}_2 \tag{4.6}$$

which can also be written as

$$\begin{aligned}
\left.\frac{d\vec{A}}{dt}\right|_{R_1} &= \dot{x}_2\vec{\imath}_2 + x_2\left.\frac{d\vec{\imath}_2}{dt}\right|_{R_1} + \dot{y}_2\vec{\jmath}_2 + y_2\left.\frac{d\vec{\jmath}_2}{dt}\right|_{R_1} + \dot{z}_2\vec{\imath}_2 + z_2\left.\frac{d\vec{k}_2}{dt}\right|_{R_1}, \\
&= \dot{x}_2\vec{\imath}_2 + x_2(\vec{\omega}\,(R_2/R_1)\wedge\vec{\imath}_2) + \dot{y}_2\vec{\jmath}_2 + y_2(\vec{\omega}\,(R_2/R_1)\wedge\vec{\jmath}_2) + \dot{z}_2\vec{\imath}_2 + z_2\left(\vec{\omega}\,(R_2/R_1)\wedge\vec{k}_2\right), \\
&= \dot{x}_2\vec{\imath}_2 + \dot{y}_2\vec{\jmath}_2 + \dot{z}_2\vec{\imath}_2 + \vec{\omega}\,(R_2/R_1)\wedge\left(x_2\vec{\imath}_2 + y_2\vec{\jmath}_2 + z_2\vec{k}_2\right), \\
&= \left.\frac{d\vec{A}}{dt}\right|_{R_2} + \vec{\omega}\,(R_2/R_1)\wedge\vec{A}
\end{aligned}$$

This allows us to write the derivative of vector $\vec{A}$ in reference frame R_1 knowing its expression in reference frame R_2 :

$$\left.\frac{d\vec{A}}{dt}\right|_{R_1} = \left.\frac{d\vec{A}}{dt}\right|_{R_2} + \vec{\omega}\,(R_2/R_1)\wedge\vec{A} \tag{4.7}$$

4.2 Composition of velocity

Let R_1 (O_1, x_1, y_1, z_1) be an absolute coordinate system and R_2 (O_2, x_2, y_2, z_2) a relative coordinate system.

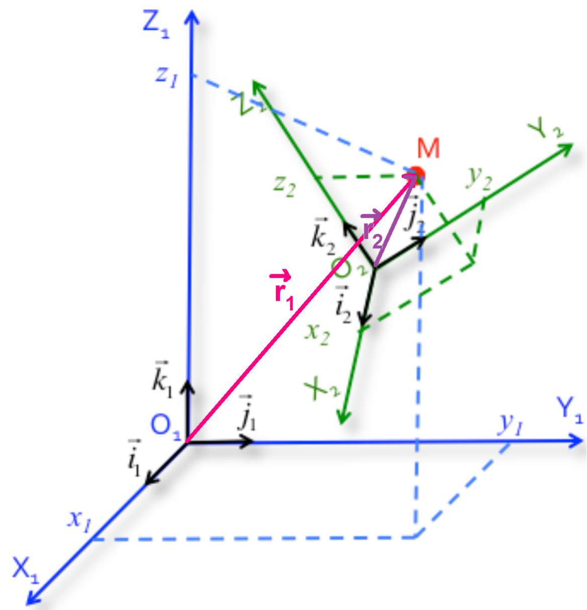


Figure 4.2 – Absolute coordinate system and relative coordinate system.

The position vectors of particle M in coordinate systems $R_1$and R_2 are, respectively:

$$\overrightarrow{O_1M} = \vec{r}_1 \qquad \text{et} \qquad \overrightarrow{O_2M} = \vec{r}_2 \tag{4.8}$$

We can write,

$$\overrightarrow{O_1M} = \overrightarrow{O_1O_2} + \overrightarrow{O_2M} \tag{4.9}$$

Therefore, the absolute velocity of point M is

$$\vec{V}_a(M) = \vec{V}\,(M/R_1) = \left.\frac{d\overrightarrow{O_1M}}{dt}\right|_{R_1} = \left.\frac{d\overrightarrow{O_1O_2}}{dt}\right|_{R_1} + \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_1}$$

$$\Rightarrow \vec{V}_a(M) = \left.\frac{d\overrightarrow{O_1O_2}}{dt}\right|_{R_1} + \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_2} + \vec{\omega}\,(R_2/R_1) \wedge \overrightarrow{O_2M}$$

where $\vec{V}\,(M/R_2) = \vec{V}_r(M) = \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_2}$ denotes the relative velocity of point M, and $\vec{V}_e(M) = \left.\frac{d\overrightarrow{O_1O_2}}{dt}\right|_{R_1} + \vec{\omega}\,(R_2/R_1) \wedge \overrightarrow{O_2M}$ is the driving velocity of M. The driving velocity of M is the absolute velocity of the (imaginary) point that coincides with M at time t and is assumed to be fixed in the reference frame $R_2\big(\vec{V}_r(M) = \vec{0}\big)$.

We can also write the driving velocity of M as follows,

$$\vec{V}_e(M) = \left.\frac{d\overrightarrow{O_1M}}{dt}\right|_{R_1} \qquad (M \text{ fixed in } R_2) \tag{4.10}$$

We therefore have,

$$\vec{V}_a(M) = \vec{V}_r(M) + \vec{V}_e(M) \tag{4.11}$$

4.3 Composition of accelerations

The absolute acceleration of point *M* is

$$\vec{\gamma}_a(M) = \vec{\gamma}\,(M/R_1) = \left.\frac{d^2\overrightarrow{O_1M}}{dt^2}\right|_{R_1} = \left.\frac{d\vec{V}_a(M)}{dt}\right|_{R_1} = \left.\frac{d\left(\vec{V}_r(M)+\vec{V}_e(M)\right)}{dt}\right|_{R_1}$$

$$\Rightarrow \vec{\gamma}_a(M) = \left.\frac{d\vec{V}_r(M)}{dt}\right|_{R_1} + \left.\frac{d\vec{V}_e(M)}{dt}\right|_{R_1}$$

$$\Rightarrow \vec{\gamma}_a(M) = \left.\frac{d\vec{V}_r(M)}{dt}\right|_{R_1} + \left.\frac{d}{dt}\left[\left.\frac{d\overrightarrow{O_1O_2}}{dt}\right|_{R_1} + \vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M}\right]\right|_{R_1} \tag{4.12}$$

with

$$\left.\frac{d\vec{V}_r(M)}{dt}\right|_{R_1} = \left.\frac{d\vec{V}_r(M)}{dt}\right|_{R_2} + \vec{\omega}\,(R_2/R_1)\wedge\vec{V}_r(M) = \vec{\gamma}_r\,(M) + \vec{\omega}\,(R_2/R_1)\wedge\vec{V}_r(M) \tag{4.13}$$

and

$$\left.\frac{d}{dt}\left[\vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M}\right]\right|_{R_1} = \left.\frac{d\vec{\omega}(R_2/R_1)}{dt}\right|_{R_1}\wedge\overrightarrow{O_2M} + \vec{\omega}(R_2/R_1)\wedge\left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_1} \tag{4.14}$$

where

$$\left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_1} = \left.\frac{d\overrightarrow{O_2M}}{dt}\right|_{R_2} + \vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M} = \vec{V}_r(M) + \vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M} \tag{4.15}$$

Therefore, the absolute acceleration can be written as

$$\vec{\gamma}_a(M) = \vec{\gamma}_r(M) + \vec{\omega}\,(R_2/R_1)\wedge\vec{V}_r(M) + \left.\frac{d^2\overrightarrow{O_1O_2}}{dt^2}\right|_{R_1} + \left.\frac{d\vec{\omega}\,(R_2/R_1)}{dt}\right|_{R_1}\wedge\overrightarrow{O_2M}$$
$$+\vec{\omega}\,(R_2/R_1)\wedge\left[\vec{V}_r(M) + \vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M}\right]$$

$$\vec{\gamma}_a(M) = \vec{\gamma}_r(M) + 2\vec{\omega}\,(R_2/R_1)\wedge\vec{V}_r(M) + \left.\frac{d^2\overrightarrow{O_1O_2}}{dt^2}\right|_{R_1} + \left.\frac{d\vec{\omega}\,(R_2/R_1)}{dt}\right|_{R_1}\wedge\overrightarrow{O_2M}$$
$$+\vec{\omega}\,(R_2/R_1)\wedge\left[\vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M}\right]$$

Of which

$$\left.\frac{d^2\overrightarrow{O_1O_2}}{dt^2}\right|_{R_1} + \left.\frac{d\vec{\omega}\,(R_2/R_1)}{dt}\right|_{R_1}\wedge\overrightarrow{O_2M} + \vec{\omega}\,(R_2/R_1)\wedge\left[\vec{\omega}\,(R_2/R_1)\wedge\overrightarrow{O_2M}\right] = \vec{\gamma}_e(M) \tag{4.16}$$

denotes the drive acceleration, and

$$2\vec{\omega}\,(R_2/R_1) \wedge \vec{V}_r(M) = \vec{\gamma}_c(M) \tag{4.17}$$

is the Coriolis acceleration or complementary acceleration. We then write

$$\vec{\gamma}_a(M) = \vec{\gamma}_r(M) + \vec{\gamma}_e(M) + \vec{\gamma}_c(M) \tag{4.18}$$

Special case:
When the reference frame R_2 is in translation relative to R_1,

$$\vec{\omega}\,(R_2/R_1) = \vec{0} \tag{4.19}$$

Consequently,

$$\vec{V}_a(M) = \vec{V}_r(M) + \vec{V}_a(O_2) \qquad \text{et} \qquad \vec{\gamma}_a(M) = \vec{\gamma}_r(M) + \vec{\gamma}_a(O_2) \tag{4.20}$$

If, in addition, R_2 is in uniform translation relative to R_1,

$$\vec{V}_a(O_2) = \overrightarrow{cste} \qquad \text{et} \qquad \vec{\gamma}_a(M) = \vec{\gamma}_r(M) \tag{4.21}$$

Chapter 5

Dynamics of a material point

Dynamics is the description of the motions of a material point, taking into account the causes that produce them. The fundamental concepts of the dynamics of a material point are summarized in Newton's three laws.

5.1 Mass and center of inertia

The **mass** of a system characterizes the amount of matter it contains. It is invariable in Newtonian mechanics. It is a characteristic of the system. In the International System of Units, the unit of mass is the kilogram (*kg*). The **center of inertia** of a material system (or center of gravity) corresponds to the point denoted *G*, the barycenter of the positions of the material points assigned their mass.

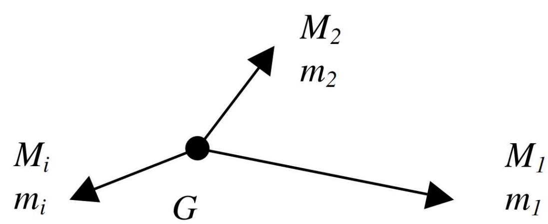


Figure 5.1 – Center of inertia of a material system.

By definition of the barycenter, point *G* satisfies:

$$\sum_i m_i \overrightarrow{GM_i} = \vec{0} \tag{5.1}$$

For a discrete system consisting of *n* masses located at points M_i, relative to a point *O* as the origin, we have:

$$\overrightarrow{OG} = \frac{\sum_i m_i \overrightarrow{OM_i}}{\sum_i m_i} \Rightarrow m.\overrightarrow{OG} = \sum_i m_i \overrightarrow{OM_i} \tag{5.2}$$

where *m* is the total mass of the system.

If the system forms a continuous medium on a macroscopic scale, the sum sign becomes an integral sign:

$$m.\overrightarrow{OG} = \iiint_M \overrightarrow{OM} dm \tag{5.3}$$

5.2 Concept of force

5.2.1 Notion of force

A material point G is rarely mechanically isolated but undergoes actions. These actions are called forces. When talking about force, it is important to note that this implies the existence of an actor (the one exerting the force) and a receiver (the one undergoing the force). A force is measured using a dynamometer and is expressed in Newton (symbol N) in the International System of Units ($1N = 1kg.m.s^{-2}$).

5.2.2 Force vector

Any force can be represented by a vector $\vec{F}$ called **a force vector**, which mathematically expresses the actions of the neighboring material point on the material point under study, and which is characterized by:

- a direction that represents the line of action of the force,
- a direction in which the action is exerted,
- a point of application that represents the point where the action is exerted on the body,
- an intensity schematized by the length of the force vector.

Forces are additive, meaning that if n forces act simultaneously on a body, the movement of the latter is the same as if it were subjected to a single force equal to the vector sum of *the n* forces. This sum is called the resultant of the n forces.

5.2.3 Classification of forces

The forces that a material point can undergo are in fact limited in number. We distinguish between the following forces:

- **Real (or external) forces**

There are two types of real forces:

- **Force at a distance**

Newtonian gravitational force: Any two material points S and C with respective masses M and m separated by a distance SC always attract each other with a force $\vec{F}$ collinear with SC. This force can be written as:

$$\vec{F}_{M\to m} = -g\frac{mM}{SC^2}\vec{u} = -g\frac{mM}{SC^3}\overrightarrow{SC} \tag{5.4}$$

where g is the universal gravitational constant; $g = 6{,}67.10^{-11}m^3.kg^{-1}.s^{-2}$.

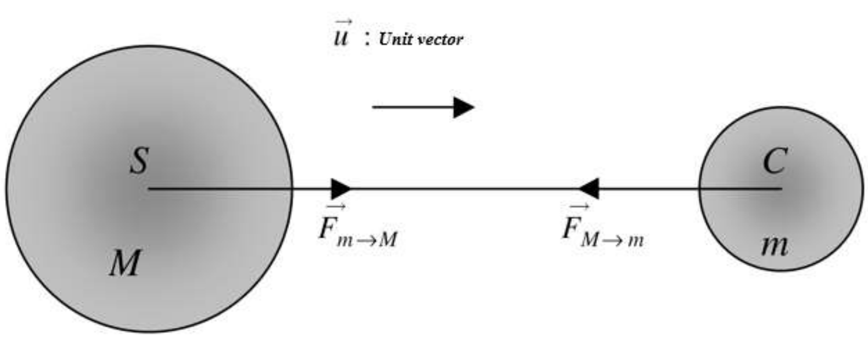


Figure 5.2 – Gravitational forces of an object of mass M on an object of mass m.

Coulomb force: The Coulomb interaction is analogous to the gravitational interaction for point electric charges. The interaction force of a charge Q placed at S on a charge q placed at C is written as:

$$\vec{F}_{Q\to q} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{SC^3}\overrightarrow{SC} \quad (5.5)$$

As in the case of gravity, it is possible to show a field created by a point charge Q at any point M in space. This field, called the electric field, is written as:

$$\vec{E}(M) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{SM^3}\overrightarrow{SM} \quad (5.6)$$

Any charge q placed in this field will undergo an action from the charge Q, which can be written as:

$$\vec{F}_{Q\to q} = q\vec{E} \quad (5.7)$$

Lorentz force: The force experienced by an electric charge placed in fields $\vec{E}$ and $\vec{B}$ is called the Lorentz force and is written as:

$$\vec{F} = q(\vec{E} + \vec{V} \wedge \vec{B}) \quad (5.8)$$

where $\vec{V}$ is the velocity vector of the charge in the reference frame where $\vec{E}$ and $\vec{B}$ are measured.

- **Contact forces**

Friction forces: Friction forces are forces that arise either when an object moves or when that object is subjected to a force that tends to move it. In all cases, the friction force opposes the movement that is being attempted. It is important to distinguish between two types of friction: viscous friction (solid-fluid contact) and solid friction (solid-solid contact).

Viscous friction: When a solid moved in a fluid (gas such as air or liquid like water), it is subjected to frictional forces from the fluid. The resultant of these actions is a force vector proportional to the velocity vector $\vec{V}$ of the object's displacement.

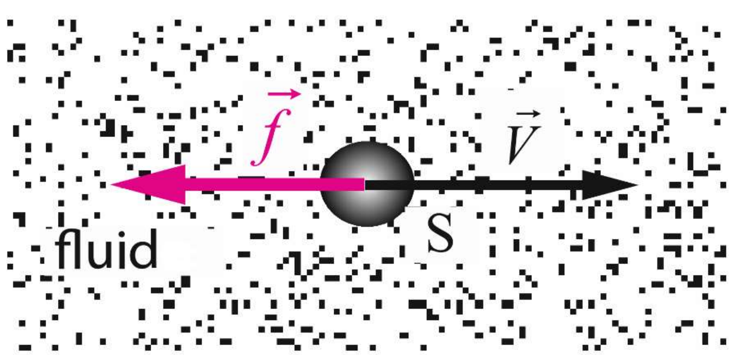


Figure 5.3 – Displacement of a solid S in a fluid.

When the velocity of displacement of *S* is low, the viscous friction force is given by:

$$\vec{f} = -k\vec{V} \tag{5.9}$$

where k is a positive coefficient depending on the viscosity of the fluid and the geometry of S.

Solid friction: Consider a solid S resting on a horizontal support and subjected to an external driving force, $\vec{F}$. Let $\vec{R}$ be the reaction of the support and $\vec{f}$ the friction force generated by the displacement of the object under the effect of $\vec{F}$. The reaction $\vec{R}$ of the support can be written as the vector sum of a component normal to the support, denoted $\vec{R}_N$, and a component tangential to the support, corresponding to the friction force $\vec{f}$:

$$\vec{R} = \vec{R}_N + \vec{f} \tag{5.10}$$

Let $\vec{P}$ be the weight of object *S*. All forces have their point of application at the center of mass of *S*, denoted *G*.

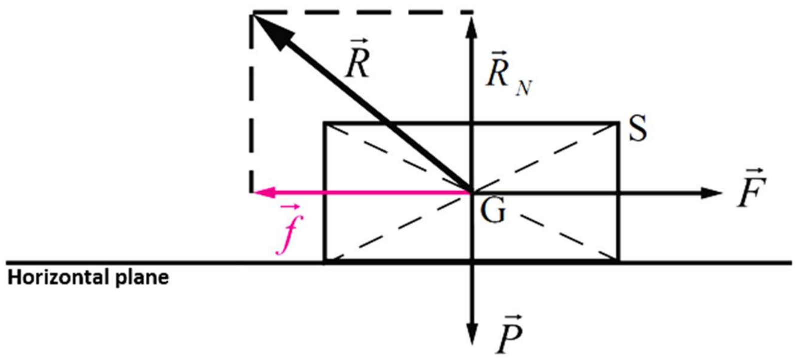


Figure 5.4 – Solid in equilibrium on a support under the action of an external force and a friction force.

Experience shows that the intensity of the friction force $\vec{f}$ is proportional to that of the normal reaction of the support $\vec{R}_N$. This proportionality factor, denoted μ, corresponds to the coefficient of friction such that:

$$\mu = \frac{\|\vec{f}\|}{\|\vec{R}_N\|} \tag{5.11}$$

Binding force: To describe the movement of a material point free of any constraints in three-dimensional space, six parameters (position and velocity) are required. We therefore say that the material point has six degrees of freedom (3 translations + 3 rotations). A material point is subject to constraints if its position (and/or) velocity must satisfy a physical constraint. Constraints therefore reduce the number of degrees of freedom of the material point.

Let us consider the example of a material point *M* with mass *m* at the end of an infinitely thin (negligible thickness) inextensible wire (pendulum).

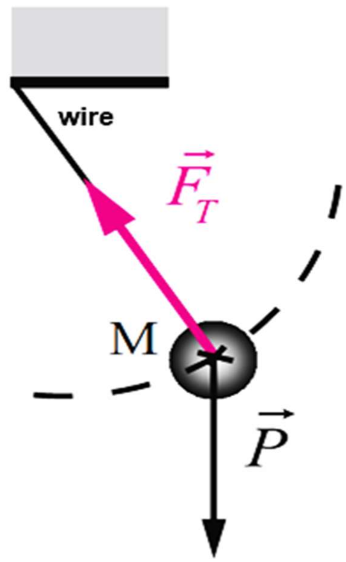


Figure 5.5 – Tension of a wire.

The wire exerts a binding force, denoted $\vec{F}_T$, on the material point *M*, restricting its movement (3 possible rotations and no translation). This force is exerted by the wire as long as it is taut, and is called the *tension of the wire*. The expression for the tension of the wire can be determined from Newton's second law.

- **Inertial (or internal) forces**

This is the resistance that bodies exhibit to movement. This resistance is due to their mass. They are,

+ **The driving inertia force:**

$$\vec{F}_e = -m.\vec{\gamma}_e \tag{5.12}$$

+ **Coriolis inertia force:**

$$\vec{F}_c = -m.\vec{\gamma}_c \tag{5.13}$$

The forces $\vec{F}_e$ and $\vec{F}_c$ only appears in Galilean reference frames.

+ **Centrifugal force:** Whenever an object moves in a circular path and remains on its trajectory, it will be affected by both:

- centripetal or radial force $\vec{F}_r$ is parallel to $\vec{\gamma}$,
- centrifugal force $\vec{F}_c$ is opposite to $\vec{\gamma}$, with:

$$F_c = F_r = m\omega^2 r \tag{5.14}$$

Centrifugation: Centrifugation is a mechanical separation process that uses centrifugal force to separate two or three phases in a rotating motion. It is one of the most interesting applications of centrifugal force.

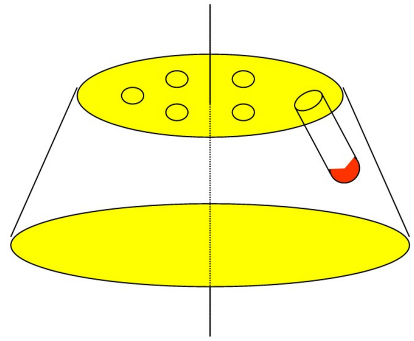


Figure 5.6 – A tube placed in a centrifuge.

Under the effect of effective weight P_{eff} (which is less than mg), a particle can settle to the bottom of a tube provided that its density is greater than that of the liquid in which it is found. If the tube is tilted to a horizontal position and rotated, the particles in the liquid will be subjected to centrifugal force and will move to the bottom of the tube: this is sedimentation under the effect of F_c. This force can be 10^6 times greater than P_{eff}. It depends mainly on the velocity of rotation. This is the physical principle behind the centrifuge. Centrifuges are used in several fields:

- isolating red blood cells from serum,
- separating precipitates or bacteria,
- separating fats (butter from milk, for example),
- sedimentation of protein molecules.
- If the solution contains several types of particles, they will be identified by their sedimentation rates, which depend on their mass. This will enable the different components of the mixture (biological solutions, etc.) to be identified.

5.3 Galilean reference frame

5.3.1 Definition

The principle of inertia also allows us to define the Galilean reference frame, which is any reference frame where the principle of inertia is applicable.
The principle of inertia therefore stipulates that the acceleration of an isolated material point is zero in a Galilean reference frame. However, according to the results of the previous chapter, the acceleration of the material point will also be zero in any reference frame moving in a straight line at a constant velocity relative to a Galilean reference frame. This leads to the following result:
Any reference frame moving in uniform rectilinear motion relative to a Galilean reference frame is also Galilean.

5.3.2 Examples of Galilean reference frames

Copernican reference frame

We saw in the previous chapter that

$$\vec{\gamma}_a(M) = \vec{\gamma}_r(M) + \vec{\gamma}_e(M) + \vec{\gamma}_c(M) \tag{5.15}$$

Consequently, the fundamental principle of dynamics will not be written in the same way in R_1 and in R_2.

We therefore base ourselves on a result from celestial mechanics which assumes that the fundamental principle of dynamics is valid in a reference system called the Copernican reference frame. This reference frame is denoted R_c (S, X_c, Y_c, Z_c).
The Copernican reference frame has its center at the center of the solar system, and its axes are given by the directions of three very distant stars that are assumed to be fixed relative to the sun.

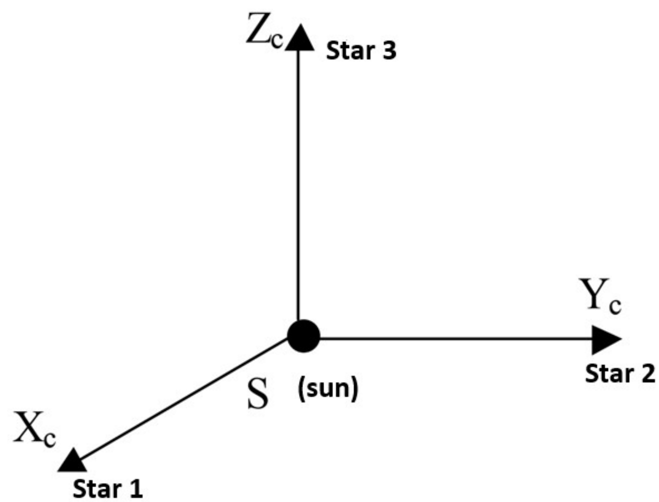


Figure 5.7 – Copernican reference frame.

Note:
If we study the motion of point M relative to reference frame R', with R' in uniform translation relative to the Copernican reference frame, the fundamental law of dynamics will also be valid in R'. Indeed,

$$\vec{\gamma}(M/R_c) = \vec{\gamma}(M/R') \tag{5.16}$$

because

$$\vec{\gamma}_e(M) = \vec{\gamma}_c(M) = \vec{0} \tag{5.17}$$

Any reference frame in uniform linear motion relative to the Copernican reference frame will be called a Galilean reference frame.

Geocentric reference frame

The geocentric reference frame has its center at the center of the Earth, and its axes have fixed directions that are those of the Copernican reference frame.

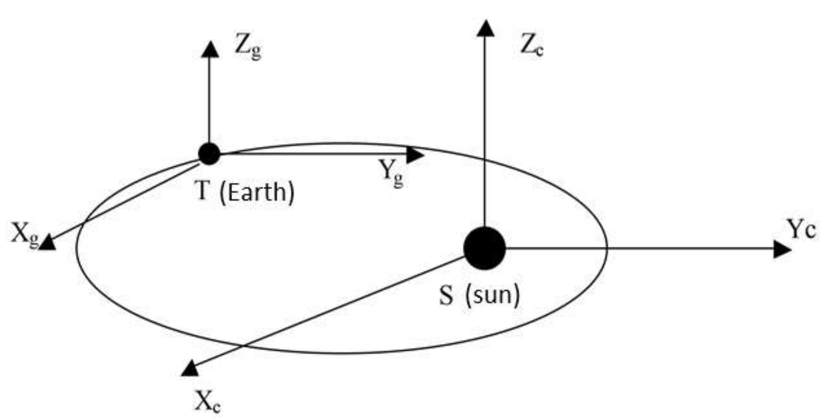


Figure 5.8 – Geocentric reference frame.

The geocentric reference frame is considered to be a Galilean reference frame for experiments whose duration is very short compared to the period of the Earth's revolution around the Sun. In fact, the Earth revolves around the Sun and, as we will see later, its motion is elliptical with a period of revolution of approximately 365 days. Its motion is therefore not uniform in a straight line relative to the Copernican reference frame. However, we can consider it to be in uniform rectilinear translation for a very short time compared to the period of the Earth's revolution around the Sun.

Terrestrial reference frame

A terrestrial reference frame is a reference frame linked to the Earth. Its origin is therefore a point on the planet and its axes are fixed relative to it.
The terrestrial reference frame rotates relative to the geocentric reference frame with a period of 24 hours. Therefore, it is not truly a Galilean reference frame. However, for physical phenomena whose duration is very short compared to 24 hours, it can be considered Galilean. It is sometimes also called *the laboratory reference frame*.

5.4 Fundamental laws of dynamics

5.4.1 Principle of inertia: Newton's first law

Principles or laws cannot be proven.

There is a class of privileged reference frames in which the motion of any free particle is uniform and rectilinear; these are called inertial or Galilean reference frames.
In a Galilean reference frame, an isolated system is either at rest or moving in a straight line at a constant velocity. A Galilean reference frame is therefore a reference frame in which the principle of inertia applies.
For isolated M (subject to no interaction) in a Galilean reference frame R:

$$\vec{V}_{M/R} = \overrightarrow{cste} \Leftrightarrow \vec{\gamma} = \vec{0} \qquad (5.18)$$

This is a special case of the fundamental relation of dynamics when $\vec{F} = \vec{0}$.

5.4.2 Fundamental principle of dynamics (FPD): Newton's second law

This law is also called the "kinetic resultant theorem" or "quantity theorem." of motion" or "center of inertia theorem."
In a Galilean reference frame R, the vector sum of the external forces $\vec{F}_{ext}$ acting on a material point M is equal to the product of the acceleration vector $\vec{\gamma}\,(M)$ and the mass m of the material point:

$$\sum \vec{F}_{ext} = m\vec{\gamma}\,(M) \qquad (5.19)$$

This law allows us to link the kinematics of a material point to the causes of motion. Thus, so-called pseudo- isolated systems (systems for which the sum of the forces applied is zero) have zero acceleration.

5.4.3 Principle of reciprocal actions: Newton's third law

When two particles $M_1$and M_2interact, regardless of the frame of reference and regardless of their motion (or lack thereof), the action $\vec{F}_{1\to2}$ of particle M_1 on particle M_2 is exactly opposite to the reaction $\vec{F}_{2\to1}$ of particle M_2 on particle M_1 :

$$\vec{F}_{1\to2} = -\vec{F}_{2\to1} \tag{5.20}$$

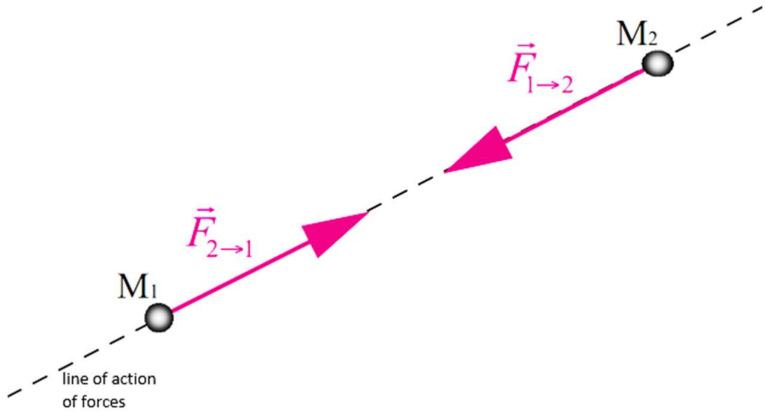


Figure 5.9 – Illustration of the principle of reciprocal actions.

It is important to note that this law, also known as the principle of action and reaction, is independent of the frame of reference.

5.5 Expression of PFD using the amount of movement

5.5.1 Definition

For a material point *M*, with mass *m*, moving in a reference frame *R*, the momentum of *M* relative to *R* is defined by:

$$\vec{p}(M/R) = m\vec{V}\,(M/R) \tag{5.21}$$

5.5.2 Acceleration vector

By deriving the above definition, we can show that the derivative of momentum is proportional to acceleration. Indeed, we have:

$$\left.\frac{d\vec{p}(M/R)}{dt}\right|_R = m\vec{\gamma}(M/R) \tag{5.22}$$

The acceleration quantity $\vec{\Gamma}\,(M)$ of point *M* relative to a reference frame *R* is defined as the product of its mass *m* and its acceleration vector $\vec{\gamma}(M/R)$:

$$\vec{\Gamma}\,(M/R) = m\vec{\gamma}(M/R) = \left.\frac{d\vec{p}(M/R)}{dt}\right|_R \tag{5.23}$$

5.5.3 Fundamental principle of dynamics

The fundamental principle of dynamics can therefore be expressed in terms of the quantity of motion:

$$\left.\frac{d\vec{p}(M/R)}{dt}\right|_R = m\vec{\gamma}(M/R) = \sum \vec{F}_{ext} \tag{5.24}$$

The derivative of momentum with respect to time is simply the resultant of the external forces and applied to particle *M* (the reference frame *R* is assumed to be Galilean).

5.6 Fundamental principle of dynamics in a non-Galilean reference frame

The statement of the fundamental principle of dynamics given above is valid in a Galilean reference frame. However, we have seen, with the law of composition of accelerations, that acceleration is not necessarily the same in all reference frames. In particular, acceleration in a Galilean reference frame is not the same as in a non-Galilean reference frame.

5.6.1 PFD and inertial forces

Consider a non-Galilean reference frame R' moving relative to a Galilean reference frame *R*. The PFD in the Galilean reference frame *R* is written as:

$$\sum \vec{F}_{ext} = m\vec{\gamma}(M/R) \tag{5.25}$$

R being the absolute reference frame and R' the relative reference frame, the law of composition of accelerations is then written as:

$$\vec{\gamma}\,(M/R) = \vec{\gamma}_a(M) = \vec{\gamma}_r(M) + \vec{\gamma}_e(M) + \vec{\gamma}_c(M) \tag{5.26}$$

The PFD in *R* then becomes,

$$\sum \vec{F}_{ext} = m\vec{\gamma}_r\,(M) + m\vec{\gamma}_e\,(M) + m\vec{\gamma}_c\,(M) \tag{5.27}$$

This allows us to write the PFD in the non-Galilean (relative) reference frame R':

$$\begin{aligned} m\vec{\gamma}(M/R') &= m\vec{\gamma}_r\,(M)\,, \\ &= \sum \vec{F}_{ext} - m\vec{\gamma}_e\,(M) - m\vec{\gamma}_c\,(M)\,, \\ &= \sum \vec{F}_{ext} + \vec{F}_e + \vec{F}_c \end{aligned} \tag{5.28}$$

where we have called the terms $-m\vec{\gamma}_e\,(M)$ and $-m\vec{\gamma}_c\,(M)$ the inertial forces. In particular, we have:

- $\vec{F}_e = -m\vec{\gamma}_e\,(M)$: is the driving inertia force.
- $\vec{F}_c = -m\vec{\gamma}_c\,(M)$: is the Coriolis inertial force.

In a non-Galilean reference frame, in addition to the external forces acting on the material point, inertial forces must also be taken into account. However, it is important to note that inertial forces are not due to any particular interaction. They are therefore not considered to be real forces in the same way as other forces, even though their physical effects are real.

Notes:

- If R' is a Galilean reference frame, the inertial forces are zero, and the PFD therefore applies without modification.
- The PFD in a non-Galilean reference frame is also expressed using momentum:

$$\left.\frac{d\vec{p}(M/R')}{dt}\right|_{R'} = \sum \vec{F}_{ext} + \vec{F}_e + \vec{F}_c \quad (5.29)$$

if R' is not Galilean.

5.6.2 Special examples

R′* in rectilinear translation relative to *R

In this case, we have:

$$\vec{\gamma}_e(M) = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R \quad \text{et} \quad \vec{\gamma}_c(M) = \vec{0} \quad (5.30)$$

This gives the inertial forces:

$$\vec{F}_e(M) = -m\left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R \quad \text{et} \quad \vec{F}_c(M) = \vec{0} \quad (5.31)$$

R′* rotating relative to *R

Consider the case where the reference frame R' is rotating relative to the absolute reference frame R, and the rotation is about an axis passing through the origin common to both reference frames O:

$$\vec{\omega}\,(R'/R) = \vec{\omega} \quad (5.32)$$

We then have:

$$\vec{\gamma}_e(M) = \left.\frac{d\vec{\omega}}{dt}\right|_R \wedge \overrightarrow{OM} + \vec{\omega} \wedge \left[\vec{\omega} \wedge \overrightarrow{OM}\right] \quad \text{et} \quad \vec{\gamma}_c(M) = 2\vec{\omega} \wedge \vec{V}_r \quad (5.33)$$

The inertial forces are obtained by multiplying these accelerations by the factor $(-m)$:

$$\vec{F}_e = -m\left.\frac{d\vec{\omega}}{dt}\right|_R \wedge \overrightarrow{OM} - m\vec{\omega} \wedge \left[\vec{\omega} \wedge \overrightarrow{OM}\right] \quad \text{et} \quad \vec{F}_c = -2m\vec{\omega} \wedge \vec{V}_r \quad (5.34)$$

If, in addition, the rotational motion of R' relative to R is uniform: $\vec{\omega} = \overrightarrow{cste}$, the inertial forces can then be written as:

$$\vec{F}_e = -m\vec{\omega} \wedge \left[\vec{\omega} \wedge \overrightarrow{OM}\right] \quad \text{et} \quad \vec{F}_c = -2m\vec{\omega} \wedge \vec{V}_r \quad (5.35)$$

5.7 Angular momentum theorem

In many cases, it is more convenient to use the angular momentum theorem than the PFD.
In what follows, we consider the motion of a material point *M* with mass *m* moving in a reference frame *R* and *O* a fixed point in this reference frame.

5.7.1 Angular momentum relative to a fixed point

The angular momentum of the material point *M* relative to *O* in the reference frame *R* is defined by:

$$\vec{\sigma}_0\,(M/R) = \overrightarrow{OM} \wedge \vec{p}(M/R) = \overrightarrow{OM} \wedge m\vec{V}(M/R) \tag{5.36}$$

5.7.2 Kinetic moment relative to an axis

The angular momentum of material point *M* relative to a fixed axis Δ, passing through *O* and with unit vector $\vec{u}_\Delta$, in reference frame *R* is defined by:

$$\mathcal{M}_\Delta\,(M/R) = \vec{\sigma}_0\,(M/R).\vec{u}_\Delta \tag{5.37}$$

5.7.3 Dynamic moment relative to a fixed point

The dynamic moment of material point *M* relative to *O* in reference frame *R* is defined by:

$$\vec{\delta}_0(M/R) = \frac{d\vec{\sigma}_0\,(M/R)}{dt} = \overrightarrow{OM} \wedge m\vec{\gamma}(M/R) \tag{5.38}$$

Indeed, the derivative of kinetic moment is:

$$\frac{d\vec{\sigma}_0\,(M/R)}{dt} = \frac{d\left(\overrightarrow{OM} \wedge m\vec{V}(M/R)\right)}{dt} = \frac{d\overrightarrow{OM}}{dt} \wedge m\vec{V}(M/R) + \overrightarrow{OM} \wedge m\frac{d\vec{V}(M/R)}{dt} \tag{5.39}$$

The first term gives:

$$\frac{d\overrightarrow{OM}}{dt} \wedge m\vec{V}(M/R) = \vec{V}(M/R) \wedge m\vec{V}(M/R) = \vec{0} \tag{5.40}$$

and the second gives:

$$\overrightarrow{OM} \wedge m\frac{d\vec{V}(M/R)}{dt} = \overrightarrow{OM} \wedge m\vec{\gamma}(M/R) \tag{5.41}$$

This ultimately gives the expression for the dynamic moment:

$$\vec{\delta}_0(M/R) = \frac{d\vec{\sigma}_0\,(M/R)}{dt} = \overrightarrow{OM} \wedge m\vec{\gamma}(M/R) \tag{5.42}$$

5.7.4 Moment of a force

If M is subjected to a force $\vec{F}$, then the moment of force $\vec{F}$ relative to point O in reference frame R is defined by:

$$\vec{\Pi}_0(\vec{F}) = \overrightarrow{OM} \wedge \vec{F} \tag{5.43}$$

The moment of force $\vec{F}$ relative to a fixed axis Δ, passing through O and with unit vector $\vec{u}_\Delta$, in reference frame R is defined by:

$$\vec{\Pi}_\Delta(\vec{F}) = \vec{\Pi}_0(\vec{F}).\vec{u}_\Delta \tag{5.44}$$

5.7.5 Theorem of angular momentum in a Galilean reference frame

In a Galilean reference frame, the dynamic moment of a material point M relative to a fixed point O in a Galilean reference frame R is equal to the moment of the resultant of the external forces exerted on M:

$$\vec{\delta}_0(M/R) = \frac{d\vec{\sigma}_0\,(M/R)}{dt} = \vec{\Pi}_0\left(\sum \vec{F}_{ext}\right) \tag{5.45}$$

The theorem of angular momentum is a direct consequence of the definition of dynamic momentum and the fundamental principle of dynamics in a Galilean reference frame:

$$\vec{\delta}_0(M/R) = \frac{d\vec{\sigma}_0\,(M/R)}{dt} = \overrightarrow{OM} \wedge m\vec{\gamma}(M/R) = \overrightarrow{OM} \wedge \sum \vec{F}_{ext} = \vec{\Pi}_0\left(\sum \vec{F}_{ext}\right) \tag{5.46}$$

5.7.6 Theorem of angular momentum projected onto an axis Δ

The kinetic momentum theorem can be expressed in relation to an axis Δ:

$$\frac{d\mathcal{M}_\Delta\,(M/R)}{dt} = \Pi_\Delta\left(\sum \vec{F}_{ext}\right) \tag{5.47}$$

Chapter 6

Work, Energy and Power

A material point and its external environment exchange work, power, energy kinetic energy, potential energy, and mechanical energy.

6.1 Work and Power of a Force

6.1.1 Work of a Force

6.1.1.1. Elementary work of a force

When moving an object, the greater the distance traveled and the greater the force applied, the greater the effort required. This effort may also depend on the trajectory followed to move the object. Work is a physical concept that takes this effort into account.

The elementary work of force $\vec{F}$ applied to material point *M* during its elementary displacement $d\overrightarrow{OM}$ is given by:

$$\delta W = \vec{F}.d\overrightarrow{OM} \tag{6.1}$$

6.1.1.2. Total work done by a force

Let *M* be a material point describing a trajectory (ℭ) relative to a reference frame R. We assume that the material point passes through point M_1at time $t_1$and through point M_2at time t_2. The work done by force $\vec{F}$ during this displacement is:

$$W_{M_1\to M_2}(\vec{F}) = \int_{M_1}^{M_2} \vec{F}.d\overrightarrow{OM} \tag{6.2}$$

The unit of work is the Joule [Joule = *N.m*].

- A force is said to be driving if its work is positive $W > 0$.
- The force is resistive if its work is negative $W < 0$.
- Force does not work if its work is zero $W = 0$.

6.1.2 Power of a force

Let us consider a material point *M* with velocity $\vec{V}$ relative to a reference frame *R*, subjected to a force $\vec{F}$. Introducing the definition of the elementary work of a force performed between times *t* and *t* + *dt*, it is possible to define instantaneous power *P(t)* by:

$$P(\vec{F}) = \frac{dW}{dt} = \vec{F}.\frac{d\overrightarrow{OM}}{dt} = \vec{F}.\vec{V}\ (M/R) \tag{6.3}$$

Power depends on the reference frame, and its unit in the International System of Units is the watt (*W*).

- If the power is positive, $P(\vec{F}) > 0$, the force is said to be driving.
- If the power is negative, $P(\vec{F}) < 0$, the force is said to be resistive.
- If the power is zero, $P(\vec{F}) = 0$, it is a force that does not do work. This is the case for a
- force perpendicular to the movement of the material point or a stationary material point.

It is therefore clear that elementary work can also be expressed in terms of the power of the force and written as:

$$dW = \vec{F}.\vec{V}\,(M/R)dt = P(\vec{F})dt \tag{6.4}$$

which leads to the following expression for the work done by a force:

$$W_{M_1 \to M_2}(\vec{F}) = \int_{M_1}^{M_2} \vec{F}.\vec{V}\,(M/R)dt = \int_{M_1}^{M_2} P(\vec{F})dt \tag{6.5}$$

6.2 Energy

6.2.1 Conservative forces—Potential energy

6.2.1.1. Definition

A force is said to be conservative if its work between two points M_1 and M_2 depends only on the starting position and the ending position. In other words, the work is independent of the path taken to go from M_1 to M_2.

- **Equivalent definition:**

A force $\vec{F}$ is said to be conservative if it derives from a potential; i.e., there exists a scalar function E_p such that:

$$\vec{F} = -\overrightarrow{grad}E_p \tag{6.6}$$

E_p is then called the potential energy of point M.

Notes:

- Potential energy is defined only to within a constant; i.e., E_p and $E_p' = E_p + C$ (where C is a constant) give rise to the same conservative force:

$$\vec{F} = -\overrightarrow{grad}E_p' = -\overrightarrow{grad}(E_p + C) = -\overrightarrow{grad}E_p \tag{6.7}$$

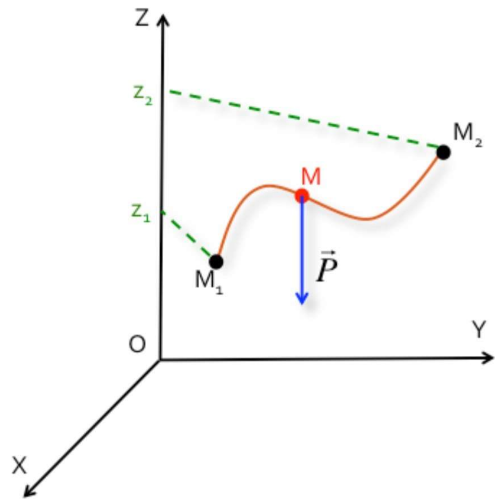
- Since $\overrightarrow{rot}(\overrightarrow{grad}f) = \vec{0}$, regardless of the function f, to verify that a force $\vec{F}$ is conservative, it suffices to verify that $\overrightarrow{rot}\vec{F} = \vec{0}$.

6.2.1.2. Examples

- **Gravitational force:**

In a Galilean reference frame $R(O, X, Y, Z)$, consider the motion of a material point M of mass m subject to Earth's gravity:

$$\vec{P} = -mg\vec{k} \tag{6.8}$$



The work done by the weight when the material point moves from M_1 to M_2 is:

$$W_{M_1\to M_2}(\vec{P}) = \int_{M_1}^{M_2} \vec{P}\,.\,d\overrightarrow{OM} = \int_{Z_1}^{Z_2} -mgdz \tag{6.9}$$

$$W_{M_1\to M_2}(\vec{P}) = -mg(z_2 - z_1) = -mg\Delta h \tag{6.10}$$

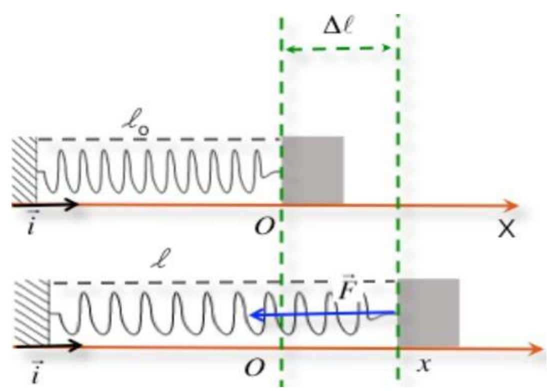
The potential energy can be calculated using the relation $\vec{P} = -\overrightarrow{grad}E_p$ and is given by:

$$E_p = mgz + C \tag{6.11}$$

where C is an integration constant.

- **Spring force:**

Consider the motion of a material point attached to a spring with stiffness k.



Based on the diagram above, the restoring force of the spring is given by:

$$\vec{F} = -k\Delta l\vec{\imath} = -kx\vec{\imath} \tag{6.12}$$

The work done by this force when the material point moves from M_1 to M_2 is:

$$W_{M_1\to M_2}(\vec{F}) = \int_{M_1}^{M_2} \vec{F}\,.\,d\overrightarrow{OM} = \int_{x_1}^{x_2} -kxdx \tag{6.13}$$

$$W_{M_1 \to M_2}(\vec{F}) = -\left(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2\right) \tag{6.14}$$

The potential energy from which the restoring force is derived is given by:

$$E_p = \frac{1}{2}kx^2 + C \tag{6.15}$$

6.2.1.3. Work done by a conservative force

From the previous examples, we can see that the work done by the force when the material point moves from M_1 to M_2 is equal to the opposite of the change in potential energy between these two positions:

$$W_{M_1 \to M_2}(\vec{F}) = -\Delta E_p = -\left(E_p(M_2) - E_p(M_1)\right) \tag{6.16}$$

- **Elementary work:**

This is a general result since we have:

$$dE_p = \overrightarrow{grad}E_p . \overrightarrow{dOM} \tag{6.17}$$

and for elementary work,

$$\delta W = \vec{F}.\overrightarrow{dOM} = -\overrightarrow{grad}E_p . \overrightarrow{dOM} \tag{6.18}$$

$$\Rightarrow \delta W = -\begin{pmatrix} \frac{\partial E_p}{\partial x} \\ \frac{\partial E_p}{\partial y} \\ \frac{\partial E_p}{\partial z} \end{pmatrix} . \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} \tag{6.19}$$

$$\delta W = -\frac{\partial E_p}{\partial x}dx - \frac{\partial E_p}{\partial y}dy - \frac{\partial E_p}{\partial z}dz \tag{6.20}$$

We therefore obtain:

$$\delta W = -dE_p \tag{6.21}$$

The elementary work can be expressed in terms of power as follows

$$\delta W = P(\vec{F})\,dt \tag{6.22}$$

which allows us to find the following relationship between the potential energy and the power of a force:

$$P(\vec{F}) = -\frac{dE_p}{dt} \tag{6.23}$$

6.2.2 Kinetic energy

Definition

For a material point of mass *m* moving at velocity $\vec{V}$ in a Galilean reference frame *R*, we will assume that the kinetic energy of this point is:

$$E_c = \frac{1}{2}m\vec{V}^2 \tag{6.24}$$

or, depending on the quantity of movement:

$$E_c = \frac{1}{2m}\vec{P}^2 \qquad (6.25)$$

Power theorem

The power of the resultant of external forces can be expressed as follows:

$$P(\vec{F}_{ext}) = \vec{F}_{ext}.\vec{V} = m\vec{\gamma}.\vec{V} = m\frac{d\vec{V}}{dt}\vec{V} \qquad (6.26)$$

where we used the PFD in a Galilean reference frame. On the other hand, we have the following relationship:

$$\frac{dV^2}{dt} = \frac{d\vec{V}^2}{dt} = \frac{d(\vec{V}.\vec{V})}{dt} = 2\frac{d\vec{V}}{dt}.\vec{V} \qquad (6.27)$$

Substituting in the first relation, we find that:

$$P(\vec{F}_{ext}) = \frac{1}{2}m\frac{dV^2}{dt} = \frac{dE_c}{dt} \qquad (6.28)$$

The power of the resultant $\vec{F}_{ext}$, of all external forces applied to a material point in a Galilean reference frame is equal to the derivative of its kinetic energy.

Kinetic energy theorem

In a Galilean reference frame, the change in kinetic energy of a material point subjected to a set of external forces between position M_1 and position M_2 is equal to the sum of the work done by these forces between these two points:

$$W_{M_1\to M_2}(\vec{F}_{ext}) = \Delta E_c = E_c(M_2) - E_c(M_1) \qquad (6.29)$$

Because,

$$\delta W = P(\vec{F})dt = \frac{dE_c}{dt}dt \Rightarrow \delta W = dE_c \qquad (6.30)$$

6.2.3 Mechanical energy

Definition

We now introduce a new function that is particularly useful in all mechanic's problems: **the mechanical energy** of a system. To define this function, we start with the kinetic energy theorem, in which we introduce the work of conservative forces $\vec{F}^{C}_{ext}$ and non-conservative forces $\vec{F}^{NC}_{ext}$, i.e.:

$$W_{M_1\to M_2}(\vec{F}_{ext}) = E_c(M_2) - E_c(M_1) = \sum W_{M_1\to M_2}(\vec{F}^{C}_{ext}) + \sum W_{M_1\to M_2}(\vec{F}^{NC}_{ext}) \qquad (6.31)$$

Calling E_p, the total potential energy, the sum of the potential energies from which each conservative force derives, we can write:

$$[E_c(M_2) - E_c(M_1)] = [E_p(M_1) - E_p(M_2)] + \sum W_{M_1\to M_2}(\vec{F}^{NC}_{ext}) \qquad (6.32)$$

which, by shifting the potential energy to the left-hand side, leads to:

$$[E_c(M_2) - E_c(M_1)] + [E_p(M_2) - E_p(M_1)] = \sum W_{M_1 \to M_2}(\vec{F}_{ext}^{NC}) \tag{6.33}$$

If we group together in the first member the functions that depend only on M_2 and M_1, we get:

$$[E_c(M_2) + E_p(M_2)] - [E_c(M_1) + E_p(M_1)] = \sum W_{M_1 \to M_2}(\vec{F}_{ext}^{NC}) \tag{6.34}$$

It is possible to introduce a new state function called mechanical energy E_m of the system by setting:

$$E_m = E_c + E_p \tag{6.35}$$

The introduction of this function allows us to present the energy balance of a system in a very simple way using the following relationship:

$$\Delta E_m = E_m(M_2) - E_m(M_1) = \sum W_{M_1 \to M_2}(\vec{F}_{ext}^{NC}) \tag{6.36}$$

which leads to the mechanical energy theorem.

Mechanical energy theorem

The change in mechanical energy of a system between two points M_1 and M_2 is equal to the sum of the work done by the non-conservative external forces applied to that system.
Since non-conservative forces are resistive forces, the mechanical energy of a system can only decrease over time. However, when a system is mechanically isolated (i.e., a system that is not subject to any external non-conservative forces), mechanical energy is conserved. Mechanical energy no longer depends on the point considered.

$$\text{Mechanically isolated system} \Leftrightarrow E_m = cste \tag{6.37}$$

or even,

$$\frac{dE_m}{dt} = 0 \tag{6.38}$$

6.3 Equilibrium and stability of a conservative system

6.3.1 Equilibrium positions

In a Galilean reference frame, consider a material point subjected to *conservative forces* whose resultant is $\vec{F}$. The equilibrium position of the material point corresponds to an *extremum* of the potential energy. Therefore, if M_0 is an equilibrium position, the first derivatives of the potential energy must be zero at this point:

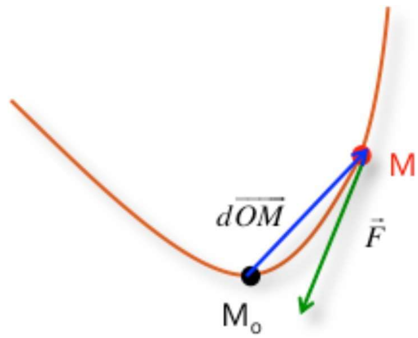
$$\left.\frac{\partial E_p}{\partial x}\right|_{M_0} = \left.\frac{\partial E_p}{\partial y}\right|_{M_0} = \left.\frac{\partial E_p}{\partial z}\right|_{M_0} = 0 \tag{6.39}$$

6.3.2 Equilibrium stability

The equilibrium position is said to be stable if the material point spontaneously returns to it following a disturbance that moves it away from this position. Otherwise, the equilibrium is unstable.

6.3.2.1 Stable Equilibrium-E_p minimal

Let M be a material point with equilibrium position M_0. M_0 is a stable equilibrium position if the potential energy is minimal at this point.



In the case of one-dimensional motion $E_p(x)$, the second derivative of the potential energy with respect to the variable x is positive in a stable equilibrium position:

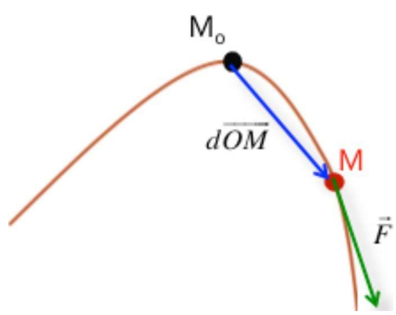
$$\left.\frac{\partial^2 E_p}{\partial x^2}\right|_{M_0} > 0 \quad (6.40)$$

i.e., point M_0 is a minimum of function $E_p\,(x)$.
The elementary work of force $\vec{F}$ when the material point is far from its equilibrium position is negative:

$$\delta W = \vec{F}.d\overrightarrow{OM} = -dE_p < 0 \quad (6.41)$$

6.3.2.2 Unstable Equilibrium-E_p maximum

M_0 is an unstable equilibrium position if the potential energy is maximal at this point.



In the case of one-dimensional motion $E_p\,(x)$, the second derivative of the potential energy with respect to the variable x is negative in an unstable equilibrium position:

$$\left.\frac{\partial^2 E_p}{\partial x^2}\right|_{M_0} < 0 \quad (6.42)$$

i.e., point M_0 is a maximum of function $E_p\ (x)$.
The elementary work of force $\vec{F}$ when the material point is far from its equilibrium position is positive:

$$\delta W = \vec{F}.d\overrightarrow{OM} = -dE_p > 0 \tag{6.43}$$

Chapter 7

Central Force Motions

7.1 Central Force

7.1.1 Definition

A material point is subject to a *central force* if this force is always directed toward a fixed point O in the reference frame under consideration.
By choosing O as the center of the reference frame, the force is therefore written as:

$$\vec{F} = F\vec{e}_r \tag{7.1}$$

$\vec{e}_r$ being the radial unit vector of the polar coordinates (noted $\vec{e}_\rho$ in Chapter 2). In this case, the central force $\vec{F}$ is parallel to the position vector Therefore, the moment of force $\vec{F}$ relative to point O is:

$$\overrightarrow{OM} \wedge \vec{F} = \vec{0} \tag{7.2}$$

Examples of central forces:

- Gravitational interaction force between two masses *m* and *M* separated by a distance *r*:

$$\vec{F} = -\frac{gmM}{r^2}\vec{e}_r \tag{7.3}$$

Where *g* denotes the universal gravitational constant.

- Electrostatic interaction force between two particles with electrostatic charges *q* and *Q* separated by a distance *r*:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{r^2}\vec{e}_r \tag{7.4}$$

where ε_0 is the permittivity of free space.

- Spring force:

$$\vec{F} = -kx\vec{\imath} \tag{7.5}$$

7.1.2 Conservation of Angular Momentum

The angular momentum of particle *M* relative to a fixed point *O*, in a reference frame *R*, is constant:

$$\left.\frac{d\vec{\sigma}_0(M/R)}{dt}\right|_R = \overrightarrow{OM} \wedge \vec{F} = \vec{0} \tag{7.6}$$

Proof:

$$\left.\frac{d\vec{\sigma}_0(M/R)}{dt}\right|_R = \left.\frac{d\left(\overrightarrow{OM}\wedge m\vec{V}\ (M/R)\right)}{dt}\right|_R = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R \wedge m\vec{V}\ (M/R) + \overrightarrow{OM}\wedge m\left.\frac{d\vec{V}\ (M/R)}{dt}\right|_R$$

$$\left.\frac{d\vec{\sigma}_0(M/R)}{dt}\right|_R = \vec{V}\ (M/R)\wedge m\vec{V}\ (M/R) + \overrightarrow{OM}\wedge m\vec{\gamma}\ (M/R) = \overrightarrow{OM}\wedge\ m\vec{\gamma}\ (M/R)$$

Using the P.F.D. of dynamics:

$$m\vec{\gamma}\ (M/R) = \vec{F} \tag{7.7}$$

and knowing that $\vec{F}$ is a central force $\left(\overrightarrow{OM}//\vec{F}\right)$ we can deduce that:

$$\left.\frac{d\vec{\sigma}_0(M/R)}{dt}\right|_R = \vec{0} \tag{7.8}$$

- Central force motions verify the law of areas. Indeed, suppose that the trajectory of particle *M* is located in the plane (*xOy*) of a reference frame *R* (*O, x, y, z*), we will have, the vector position $\overrightarrow{OM} = r\vec{e}_r$, the velocity vector $\vec{V}\ (M/R) = \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi$ and the kinetic momentum vector $\vec{\sigma}_0(M/R) = mr^2\dot{\varphi}\vec{k}$. The area constant is then written as,

$$C = r^2\dot{\varphi} \tag{7.9}$$

- The kinetic energy of a particle *M* subjected to a central force is:

$$E_c(M) = \frac{1}{2}mV^2 = \frac{1}{2}mC^2\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] \tag{7.10}$$

with $u = \frac{1}{r}$. This is *Binet's first formula.*

- The force exerted on a particle is:

$$\vec{F} = m\vec{\gamma}\ (M/R) = -mC^2u^2\left[\frac{d^2u}{d\varphi^2} + u\right]\vec{e}_r \tag{7.11}$$

This *is Binet's second formula*.

7.2 Newtonian field

7.2.1 Definition

A force is said to be Newtonian if it is a central force that varies according to the law $1/r^2$:

$$\vec{F} = -\frac{k}{r^2}\vec{e}_r \tag{7.12}$$

k being a constant. The force is attractive if *k* is positive; it is repulsive if *k* is negative.

Examples:

- Gravitational interaction: $k = gmM$
- Electrostatic interaction (Coulomb force): $k = -\frac{1}{4\pi\varepsilon_0} qQ$

7.2.2 Equation of the Trajectory

The differential equation of the motion of a material point subjected to a central force is written as:

$$\frac{d^2u}{d\varphi^2} + u = \frac{k}{mC^2} \tag{7.13}$$

This equation can be established using Binet's formulas with the fundamental principle of dynamics or by using the conservation of mechanical energy.

- **Proof 1 using the PFD:**

A Newtonian force is written as:

$$\vec{F} = -\frac{k}{r^2}\vec{e}_r = -ku^2\vec{e}_r \tag{7.14}$$

Where $u = \frac{1}{r}$. On the other hand, using Binet's second formula, we write the force as:

$$\vec{F} = -mC^2u^2\left[\frac{d^2u}{d\varphi^2} + u\right]\vec{e}_r \tag{7.15}$$

By equating the two expressions, we obtain:

$$-ku^2\vec{e}_r = -mC^2u^2\left[\frac{d^2u}{d\varphi^2} + u\right]\vec{e}_r$$

$$\Rightarrow \ k = mC^2\left[\frac{d^2u}{d\varphi^2} + u\right]$$

or even

$$\frac{d^2u}{d\varphi^2} + u = \frac{k}{mC^2} \tag{7.16}$$

- **Proof 2 using conservation of E_m:**

A Newtonian force is a conservative force, derived from potential energy, which can be written as (using $\vec{F} = -\overrightarrow{grad}E_p$) :

$$E_p = -\frac{k}{r} + Cte \tag{7.17}$$

Considering that potential energy is zero at infinity, we obtain:

$$E_p = -\frac{k}{r} = -ku \tag{7.18}$$

On the other hand, kinetic energy is written using Binet's first formula:

$$E_c = \frac{1}{2}mC^2\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] \tag{7.19}$$

Mechanical energy is then written as:

$$E_m = E_p + E_c = -ku + \frac{1}{2}mC^2\left[\left(\frac{du}{d\varphi}\right)^2 + u^2\right] \tag{7.20}$$

Since $\vec{F}$ is conservative, mechanical energy must be conserved:

$$\frac{dE_m}{dt} = 0$$

$$\Rightarrow \; -k\frac{du}{dt} + \frac{1}{2}mC^2\left[\left(\frac{d\left(\frac{du}{d\varphi}\right)^2}{dt}\right) + \frac{du^2}{dt}\right]$$

or even

$$-k\frac{du}{d\varphi}\frac{d\varphi}{dt} + \frac{1}{2}mC^2\left[2\frac{du}{d\varphi}\frac{d\left(\frac{du}{d\varphi}\right)}{dt} + 2u\frac{du}{dt}\right] = 0$$

$$-k\frac{du}{d\varphi}\frac{d\varphi}{dt} + \frac{1}{2}mC^2\left[2\frac{du}{d\varphi}\frac{d\left(\frac{du}{d\varphi}\right)}{d\varphi}\frac{d\varphi}{dt} + 2u\frac{du}{d\varphi}\frac{d\varphi}{dt}\right] = 0$$

simplifying by $\frac{du}{d\varphi}\frac{d\varphi}{dt}$ which cannot be zero:

$$-k + mC^2\left[\frac{d\left(\frac{du}{d\varphi}\right)}{d\varphi} + u\right] = 0 \tag{7.21}$$

which allows you to write (knowing that $\frac{d\left(\frac{du}{d\varphi}\right)}{d\varphi} = \frac{d^2u}{d\varphi^2}$)

$$\frac{d^2u}{d\varphi^2} + u = \frac{k}{mC^2} \tag{7.22}$$

The solution to the differential equation (second order with second member) of motion is written as:

$$u(\varphi) = u_0 cos(\varphi - \varphi_0) + \frac{k}{mC^2} \tag{7.23}$$

Using the following notation:

$$\frac{p}{\varepsilon} = \frac{mC^2}{k} \quad , \qquad e = pu_0 \tag{7.24}$$

where ε denotes the sign of k, i.e. $\begin{cases} \varepsilon = 1 & Si \;\; k > 0 \\ \varepsilon = -1 & Si \;\; k < 0 \end{cases}$, we obtain the expression of the equation of the trajectory in terms of polar coordinates (r, φ) :

$$r(\varphi) = \frac{p}{\varepsilon + e\,cos(\varphi - \varphi_0)} \tag{7.25}$$

This is the equation of a conic section with parameter p and eccentricity e, where O is one of the foci. Throughout the rest of this section, we will take $\varphi_0 = 0$ and $\varepsilon = 1$ (attractive force), giving the equation of the trajectory as:

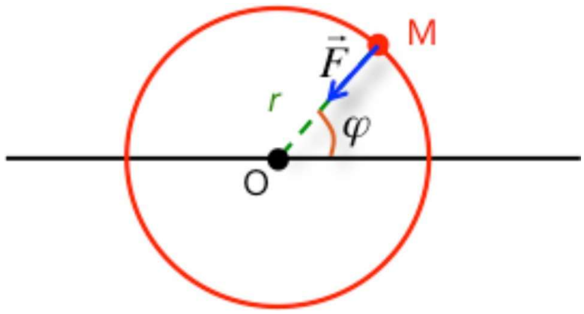
$$r(\varphi) = \frac{p}{1 + e\cos\varphi} \tag{7.26}$$

7.2.3 Classification of a Trajectory according to its Eccentricity

Depending on the value of the eccentricity e, several types of trajectories can be obtained.

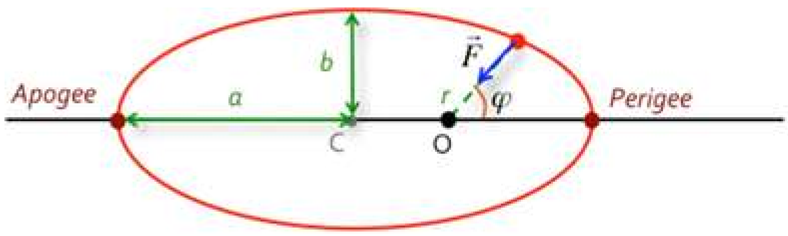
7.2.3.1 Circular trajectory

For $e = 0$, the conic is a circle, since in this case $r = p$ is constant.



7.2.3.2 Elliptical trajectory

For $0 < e < 1$, the trajectory is an ellipse, for which O is one of the foci.



The point on the trajectory closest to the focus O is called the perigee; it is obtained for the angle $\varphi = 0$, and is located at a distance r_{min} from O:

$$r_{min} = \frac{p}{1 + e} \tag{7.27}$$

In the same way, we define the apogee, which is the point on the trajectory furthest from the point O; it is obtained for $\varphi = \pi$, and is located at a distance r_{max} from O:

$$r_{max} = \frac{p}{1 - e} \tag{7.28}$$

It is important not to confuse point O, one of the foci, with the center C of the ellipse. The distance between a focus and the center is given by the following distance c:

$$c = CO = CF = \sqrt{a^2 - b^2} \tag{7.29}$$

where a is the semi-major axis and b is the semi-minor axis of the ellipse. The eccentricity e and the parameter p of the ellipse are then given by the following relations:

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{c}{a} \quad ; \quad p = \frac{b^2}{a} \quad ; \quad p = a(1 - e^2) \tag{7.30}$$

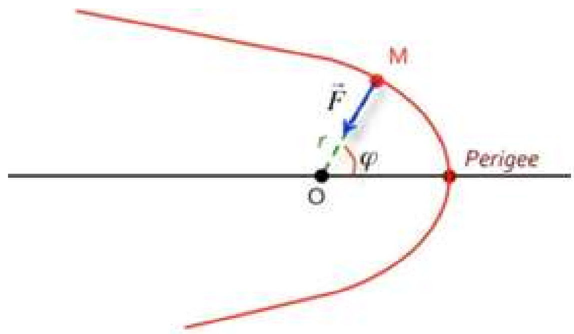
7.2.3.3 Parabolic trajectory

For $e = 1$, the trajectory is a parabola; in this case, the equation of the trajectory is written as:

$$r(\varphi) = \frac{p}{1 + cos\varphi} \tag{7.31}$$

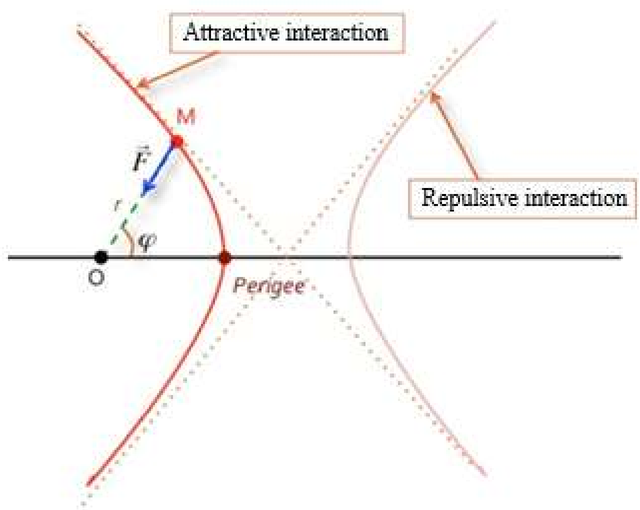
The perigee is obtained for $\varphi = 0$, and is located at a distance r_p from O:

$$r_{min} = \frac{p}{2} \tag{7.32}$$



7.2.3.4 Hyperbolic trajectory

For $e > 1$, the trajectory is a hyperbola. However, since the two branches of the hyperbola are disconnected, the material point moves only on one of the branches of the hyperbola. One corresponds to the trajectory of a material point under the action of an attractive force and the other under the action of a repulsive force.



The perigee is obtained for $\varphi = 0$, and is located at a distance r_p from O: $r_p = \frac{p}{2}$.

It should also be noted that for $e < 1$, the trajectory is closed (circle or ellipse), in which case we refer to bound states. The trajectory is open (parabola or hyperbola) for $e \geq 1$, in which case we refer to free.

7.2.4 Classification of a Trajectory according to its Mechanical Energy

7.2.4.1 Potential Energy

Since Newtonian force is conservative, it derives from potential energy:

$$\vec{F} = -\overrightarrow{grad} E_p \quad (7.33)$$

In this case, the potential energy is written as:

$$E_p = -\frac{k}{r} + Cte \quad (7.34)$$

In general, we take as the reference state for the potential energy associated with a Newtonian force when $r \to \infty$; this allows us to cancel the integration constant:

$$E_p = -\frac{k}{r} \quad (7.35)$$

Replacing r with the expression obtained for the trajectory equation E_p is then written as:

$$E_p = -\frac{k}{p}(1 + e\,cos\varphi) \quad (7.36)$$

7.2.4.2 Kinetic Energy

We use Binet's second formula:

$$E_c = \frac{1}{2} mC^2 \left[\left(\frac{du}{d\varphi} \right)^2 + u^2 \right] \quad (7.37)$$

With $u = \frac{1}{r} = \frac{1}{p}(1 + e\,cos\varphi)$ et $\frac{du}{d\varphi} = -\frac{e}{p} sin\varphi$. The kinetic energy can then be written as:

$$E_c = \frac{mC^2}{2p^2}[1 + e^2 + 2e\,cos\varphi] \quad (7.38)$$

or even

$$E_c = \frac{k}{2p}[1 + e^2 + 2e\,cos\varphi] \quad (7.39)$$

7.2.4.3 Mechanical Energy

Using the expressions established above for potential energy and mechanical energy, we obtain the expression for mechanical energy:

$$E_m = E_c + E_p,$$

$$E_m = -\frac{k}{2p}[1 - e^2]$$

Mechanical energy, as expected, is constant; its value is therefore entirely determined by the initial conditions:

$$E_m = -\frac{k}{r_0} + \frac{1}{2} mV_0^2 \quad (7.40)$$

7.2.4.4 Classification of trajectories according to mechanical energy

Mechanical energy can be used to determine the trajectory:

- $E_m = -\frac{k}{2p} < 0$, a circle ($e = 0$).
- $-\frac{k}{2p} < E_m < 0$, an ellipse ($0 < e < 1$).

$E_m = 0$, a hyperbole ($e = 1$).

$E_m > 0$, a hyperbole ($e > 1$).

Thus, we obtain a bound state (closed trajectory) for $E_m < 0$, while we have a free state (open trajectory) for $E_m \geq 0$.

7.3 Kepler's laws

Kepler's three laws are empirical laws, established from astronomical observations of planetary motion.

7.3.1 Kepler's first law

The motion of a material point M is periodic with period T.

7.3.2 Kepler's Second Law

The radius vector in the case of a central force motion sweeps equal areas during equal time intervals:

$$S = \frac{C}{2}t + S_0 \tag{7.41}$$

7.3.3 Kepler's third law

The ratio between the square of the period T of a planet's revolution around the sun and the cube of the semi-major axis a of its orbit is independent of the planet.
The areal velocity of point M is:

$$A = \frac{dS}{dt} = \frac{1}{2}C = \frac{\Delta S}{T} = \frac{\pi ab}{T} \qquad (avec\ \Delta S = S - S_0) \tag{7.42}$$

Therefore

$$A^2 = \left(\frac{\pi ab}{T}\right)^2 = \frac{C^2}{4} \tag{7.43}$$

a and b are respectively the semi-minor and semi-major axes of the ellipse. Taking into account $p = \frac{mC^2}{k}$, we write,

$$A^2 = \frac{p}{4}\frac{k}{m} = \left(\frac{\pi ab}{T}\right)^2 \tag{7.44}$$

Given that,

$$p = \frac{b^2}{a} \tag{7.45}$$

we will have,

$$T^2 = \left(\frac{4m\pi^2}{k}\right) a^3 \tag{7.46}$$

This is *Kepler's third law*. Therefore, the square of the period T^2 is proportional to the cube of the semi-major axis of the ellipse. In other words,

$$\frac{T^2}{a^3} = \frac{4m\pi^2}{k} = \frac{4\pi^2}{gM_{Sun}} = Cte \tag{7.47}$$

where g is the universal gravitational constant, and M_{Sun} represents the mass of the sun.

7.4 Artificial satellites

Consider the motion of a satellite of mass m around the Earth. Hereinafter, we will denote M_T the mass of the Earth and R_T its radius. In this case, the constant k is written as:

$$k = gmM_T \tag{7.48}$$

The motion of the satellite can be described by its mechanical energy, which is conserved:

$$E_m = -\frac{gmM_T}{r_0} + \frac{1}{2}mV_0^2 \tag{7.49}$$

or even

$$E_m = -\frac{k}{2p}[1-e^2] = -\frac{gmM_T}{2p}[1-e^2] \tag{7.50}$$

By setting the initial conditions (i.e., for a given r_0, we set a corresponding initial velocity), we determine the nature of the trajectory according to the value of the mechanical energy obtained.

7.4.1 First Cosmic Velocity-Circular Velocity

The circular trajectory of the satellite corresponds to $e = 0$ and $p = r_0$. Using the two expressions for mechanical energy, we establish the initial velocity $V_0 = V_c$, called *the first cosmic velocity*, which allows us to obtain this trajectory:

$$E_m = -\frac{gmM_T}{r_0} + \frac{1}{2}mV_c^2 = -\frac{gmM_T}{2r_0}$$

$$\Rightarrow \quad V_c = \sqrt{\frac{gM_T}{r_0}} \tag{7.51}$$

A satellite launched at an initial velocity equal to the first cosmic velocity, at a distance r_0 from the center of the Earth, will have a circular trajectory with radius r_0.

7.4.2 Second Cosmic Velocity -Escape Velocity

The escape velocity, also known as the second cosmic velocity, corresponds to the minimum initial velocity required to free the satellite from the gravitational pull of the Earth, i.e., allowing the satellite to have an open trajectory.
Consider a spacecraft of mass m such that its mechanical energy E_m is:

$$E_m = -\frac{gmM_T}{r_0} + \frac{1}{2}mV_0^2 \tag{7.52}$$

where M_T denotes the mass of the Earth and r_0 is the distance from the Earth to the spacecraft. On the other hand, the mechanical energy is written as

$$E_m = -\frac{k}{2p}[1 - e^2] \qquad with\ k = gmM_T \tag{7.53}$$

We recall that

- If $E_m < 0$, the trajectory of the spacecraft is circular or elliptical ($0 < e < 1$).
- If $E_m > 0$, the satellite's trajectory is hyperbolic ($e > 1$).
- If $E_m = 0$ ($e = 1$), the satellite's trajectory is parabolic. This corresponds to an initial velocity V_0 such that:

$$V_0 = \sqrt{\frac{2gM_T}{r_0}} = V_l \tag{7.54}$$

V_l is called the satellite's escape velocity. This velocity depends on the satellite's altitude and the radius of the Earth. Consequently,

- if the initial velocity of the spacecraft is greater than or equal to its escape velocity, its trajectory is parabolic or hyperbolic and therefore it moves away from the Earth indefinitely.
- if $0 < V_0 < V_l$, the satellite's trajectory is closed. It is circular or elliptical.

Examples:

a) At ground level: $r_0 \approx 6400km$, therefore $V_l \approx 11.2km/s$.

b) At the lunar surface: $r_0 \approx 1700km$, which corresponds to $V_l \approx 2.4km/s$.

This latter velocity is comparable to the velocity of thermal agitation of gas molecules, which explains the absence of atmosphere at the moon.

Application:

The minimum trajectory that a satellite can have corresponds to a circular trajectory at an altitude that is negligible compared to the radius of the Earth ($r_0 \approx R_T$). It corresponds to a first cosmic velocity:

$$V_c = \sqrt{\frac{gM_T}{R_T}} \tag{7.55}$$

On the other hand, the release velocity is equal to

$$V_l = \sqrt{\frac{2gM_T}{R_T}} = \sqrt{2}V_c \tag{7.56}$$

Therefore, to avoid losing a satellite, it must be launched with an initial velocity V_0 such that:

$$V_c < V_0 < V_l = \sqrt{2}V_c \tag{7.57}$$

Numerical application:

Let

$$g = 6.67 \times 10^{-11}\ N{\cdot}m^2/kg^2 \quad ; \quad M_T = 6 \times 10^{24}\ kg \quad ; \quad R_T = 6400km.$$

This gives the following numerical values for cosmic velocities:

$$V_c = 7.9 \times 10^3\ m/s = 28.5 \times 10^3\ km/h \quad ; \quad V_l = 11.3 \times 10^3\ m/s = 40.7 \times 10^3\ km/h$$

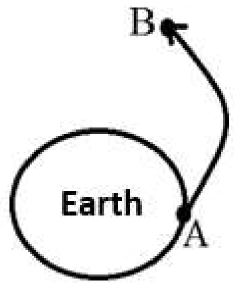
$$28.5 \times 10^3 km/h < V_0 < 40.7 \times 10^3\ km/h$$

7.4.3 Satellite launch into orbit

This is a two-stage operation:

1) *Launch from ground station A:*

At A, the launch takes place at a velocity $0 < V_0 < V_l$ (this is the ballistic phase).

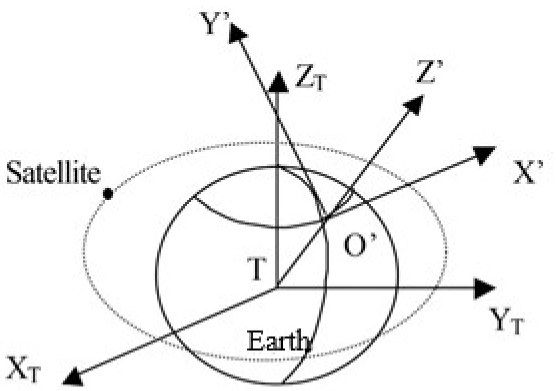


2) *Satellite deployment* (placement into orbit) takes place at B thanks to a second acceleration that will provide the necessary increase in velocity. B is generally the perigee of the ellipse.

7.4.4 Geostationary satellites

Definition

A geostationary satellite is a satellite that is fixed for a reference frame (observer) linked to the Earth. It is a satellite that has the same rotation period as the Earth, i.e., *24 hours* or *86,400 seconds*.



For the satellite to have a constant velocity, its trajectory must be circular (otherwise, as we have seen, the velocity depends on the distance from Earth, so we will have a variable velocity). We therefore use the first cosmic velocity:

$$V = V_c = \sqrt{\frac{gM_T}{R}} \tag{7.58}$$

Now, the angular velocity is given by $\omega = \frac{V}{R}$, and the period of rotation by:

$$T = \frac{2\pi}{\omega} = 2\pi\frac{R}{V} = 2\pi\sqrt{\frac{R^3}{gM_T}} \tag{7.59}$$

The radius of the trajectory of a geostationary satellite must therefore be:

$$R = \left(\frac{gM_T T^2}{4\pi^2}\right)^{\frac{1}{3}} \tag{7.60}$$

Numerical application

Let

$g = 6.67 \times 10^{-11} N\cdot m^2/kg^2$; $M_T = 6 \times 10^{24}\ kg$; $R_T = 6400km$.

$$R = 42300km = 6.6R_T$$

This corresponds to an altitude of: $h = R - R_T \approx 36000km$.

Note

Do not confuse a geostationary satellite with a geosynchronous satellite. The latter has the same rotation period as the Earth but is not fixed in relation to it. For an observer on Earth, this satellite returns to the same point in space after a period of *24 hours*.

Chapter 8

Collisions between two particles

8.1 Definition

A collision between two particles is any interaction that causes a sudden and finite change in the velocity vectors of the two particles over a very short period of time.

8.2 Conservation of momentum

Let m_1 and m_2 be the respective masses of particles M_1 and M_2 in a Galilean reference frame R_0, and let

- $\vec{V}_1$ and $\vec{V}_2$ the respective velocitys of M_1 and M_2 in the reference frame R_0 before the collision.
- $\vec{V}_1'$ and $\vec{V}_2'$ the respective velocities of M_1 and M_2 in the reference frame R_0 after the collision.
- $\vec{F}_{2\to1}$ and $\vec{F}_{1\to2}$ are the transient forces applied respectively to M_1 and M_2 only during the collision.

The reaction forces $\vec{F}_{2\to1}$ and $\vec{F}_{1\to2}$that appear during the impact are very significant, compared to the external forces applied to M_1 and M_2 .

8.2.1 Fundamental assumption

We will assume that the forces $\vec{F}_{2\to1}$ and $\vec{F}_{1\to2}$ satisfy the principle of action and reaction:

$$\vec{F}_{2\to1} + \vec{F}_{1\to2} = \vec{0} \tag{8.1}$$

Where from

$$\vec{F}_{2\to1} + \vec{F}_{1\to2} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = \frac{d(\vec{p}_1 + \vec{p}_2)}{dt} = \vec{0} \tag{8.2}$$

Therefore

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_1 + \vec{p}_2 = \overrightarrow{cte} \tag{8.3}$$

with

$$\vec{p}_1 = m_1\vec{V}_1, \quad \vec{p}_2 = m_2\vec{V}_2, \quad \vec{p}_1 = m_1\vec{V}_1' \quad et \quad \vec{p}_2 = m_2\vec{V}_2' \tag{8.4}$$

Equation (8.3) shows that the momentum of the system (S), formed by M_1 and M_2 , is conserved (the momentum of the system is the same before and after the collision).

8.2.2 Note

At the moment of collision between particles M_1 and M_2, the forces external to the system (S) are generally negligible compared to the internal forces within this system, which are the contact forces. (S) can therefore be considered an isolated system. In the case of an isolated system, the fundamental principle of dynamics applied to it in the reference frame R_0 is as follows:

$$\sum \vec{F}_{ext} = m\vec{\gamma}(S) = (m_1 + m_2)\,\vec{\gamma}(S) = \frac{d\vec{p}}{dt} = \vec{0} \tag{8.5}$$

Therefore

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = \vec{p}_1 + \vec{p}_2 = \overrightarrow{cte} \tag{8.6}$$

8.3 Elastic and inelastic collisions

8.3.1 Elastic collisions

By definition, the collision between two particles M_1 and M_2 is said to be perfectly elastic if the kinetic energy of the system (S) of the two particles before the collision is equal to the total kinetic energy of this system after the collision. We then have:

$$\frac{1}{2}m_1\vec{V}_1^2 + \frac{1}{2}m_2\vec{V}_2^2 = \frac{1}{2}m_1\vec{V}_1'^2 + \frac{1}{2}m_2\vec{V}_2'^2 \tag{8.7}$$

8.3.2 Inelastic collision

In this case, the kinetic energy of the system is not conserved during the collision. The energy balance is written as:

$$\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 + U \tag{8.8}$$

where U is the change in kinetic energy of the system (S) after the collision.

- If $U < 0$, the system absorbs energy (the impact is endoenergetic).
- If $U > 0$, the system releases energy (the impact is exoenergetic).

8.3.3 Soft impact

- Before the collision, particle M_1 has mass m_1 and velocity $\vec{V}_1$ and particle M_2 has mass m_2 and velocity $\vec{V}_2$.
- After the collision, the two particles M_1 and M_2 form a single body with mass ($m_1 + m_2$) and velocity $\vec{V}$.

In this case, the conservation of momentum is expressed as,

$$m_1\vec{V}_1 + m_2\vec{V}_2 = (m_1 + m_2)\vec{V} \tag{8.9}$$

During the collision, kinetic energy is not conserved. Losses occur in the form of heat, deformation, etc.

8.3.4 Coefficient of restitution

The coefficient of restitution (or elasticity) e is a number between 0 and 1, defined by the ratio of the relative velocities of particle M_2 to particle M_1 (or of M_1 to M_2) after the collision, i.e.:

$$e = \left| \frac{\vec{V}_1' - \vec{V}_2'}{\vec{V}_1 - \vec{V}_2} \right| \tag{8.10}$$

- If $e = 0$, the impact is soft.
- If $e = 1$, the collision is elastic.
- If $0 < e < 1$, the collision is inelastic (or intermediate).

8.4 Examples of elastic impacts

Consider a system (S) of two particles $M_1$and M_2 with masses $m_1$and m_2, respectively, whose velocities $\vec{V}_1$ and $\vec{V}_2$ before the collision become $\vec{V}_1'$ and $\vec{V}_2'$ after the collision. The conservation of momentum (vector equation) and the conservation of kinetic energy of the system (S) give four scalar equations for the six components of the unknown velocities. The six unknowns are generally the six components of the velocities $\vec{V}_1'$ and $\vec{V}_2'$. Additional information must therefore be provided in order to have as many unknowns as scalar equations.

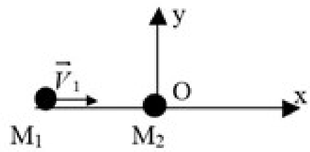
8.4.1 Direct elastic collision between two particles

A collision between two particles M_1 and M_2 is called *direct*, *head-on*, or *full-on* if the velocities before and after the collision, $\vec{V}_1$, $\vec{V}_2$, $\vec{V}_1'$ and $\vec{V}_2'$ are collinear. In this case, we have two equations with two unknowns:

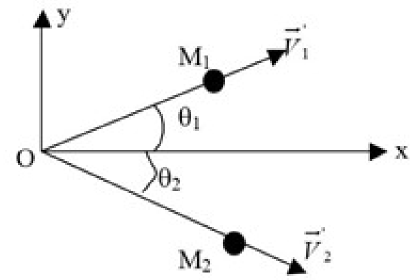
$$\begin{cases} m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2' \\ \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \end{cases} \tag{8.11}$$

8.4.2 Billiard ball collision

Suppose that particle M_1 has a velocity $\vec{V}_1$ just before the collision, in the Galilean reference frame R_0 , and that particle M_2 is at rest. We say that particles M_1 and M_2 undergo an elastic billiard ball-type collision if, after the collision, their respective velocities form angles θ_1 and θ_2 with the direction of $\vec{V}_1$.



Before the collision



After the collision

- Conservation of momentum and kinetic energy of the system (*S*) before and after the collision:

$$\begin{cases} m_1\vec{V}_1 = m_1\vec{V}'_1 + m_2\vec{V}'_2, \\ \frac{1}{2}m_1V_1^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \end{cases} \tag{8.12}$$

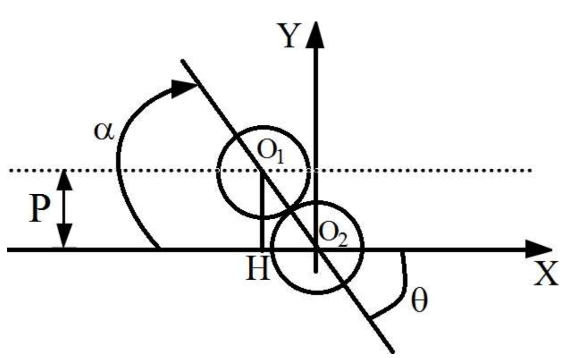
Projecting the above vector equation onto the axes of R_0 gives:

$$\begin{cases} m_1V_1 = m_1V'_1\cos\theta_1 + m_2V'_2\cos\theta_2, \\ 0 = m_1V'_1\sin\theta_1 - m_2V'_2\sin\theta_2, \\ \frac{1}{2}m_1V_1^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \end{cases} \tag{8.13}$$

We therefore have three equations for four unknowns V'_1, V'_2 , θ_1, and θ_2). To obtain the number of equations needed to find the unknowns, we introduce the impact parameter *P*. The parameter *P* is the distance separating, at the moment of impact, the center of particle M_1 from the axis (*Ox*).
The impact parameter *P* is given by:

$$P = O_1H = 2r\sin\alpha = 2r\,|\sin\theta| \tag{8.14}$$

r is the radius of M_1 and M_2 are identical. The data for *P* allows us to solve the problem of the number of unknowns.



Chapter 9

Harmonic oscillators

9.1 Free oscillators

9.1.1 Definition

A harmonic oscillator is any mechanical system whose position $q(t)$, velocity $\frac{dq(t)}{dt}$ and acceleration $\frac{d^2q(t)}{dt^2}$ are sinusoidal functions of time.
The variable $q(t)$ obeys the relation:

$$\frac{d^2q(t)}{dt^2} + \omega_0^2 q(t) = 0 \tag{9.1}$$

This is a second-order linear differential equation with constant coefficients and no second member. Its characteristic equation is:

$$r^2 + \omega_0^2 = 0 \tag{9.2}$$

The solution to this equation is of the form:

$$q(t) = A\sin(\omega_0 t + \varphi) \qquad \text{or} \qquad q(t) = A\cos(\omega_0 t + \varphi) \tag{9.3}$$

where A, ω_0 and φ are, respectively, the amplitude, frequency, and phase of the oscillation. A and φ are determined from the initial conditions. The period of the oscillation is defined by:

$$T = \frac{2\pi}{\omega_0} \tag{9.4}$$

and the frequency by:

$$f = \frac{1}{T} \tag{9.5}$$

9.1.2 Mass-spring system

9.1.2.1. Resting mass

Consider a spring with a mass that is negligible compared to the mass m attached to it.
At equilibrium, the fundamental principle of dynamics applied to mass m is written as:

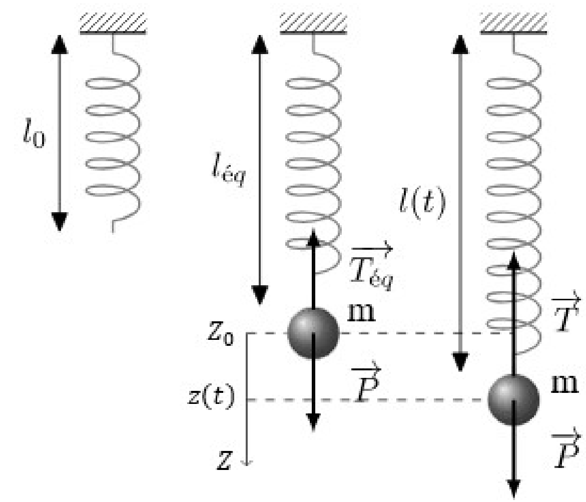
$$\vec{P} + \vec{T} = \vec{0} \tag{9.6}$$

where $\vec{P}$ is the weight of the mass m and $\vec{T}$ is the restoring force of the spring. The projection of the above vector equation onto the Oz axis gives:

$$mg + T = 0 \tag{9.7}$$

with $T = -k\,(z_0 - l_0)$ (Hooke's law). l_0 is the length of the spring when unloaded, and $z_0$is its length at equilibrium. k is the spring constant (or elasticity constant). Where from

$$mg - k(z_0 - l_0) \tag{9.8}$$



9.1.2.2. Moving mass

In this case, the mass m will be marked relative to the axis (Oz) by $z\,(t)$. The equation of motion of the mass m is:

$$mg - k(z(t) - l_0) = m\ddot{z}(t) \tag{9.9}$$

which can also be written as:

$$mg - k(z(t) - z_0 + z_0 - l_0) = m\ddot{z}(t)\,,$$

$$mg - k(z_0 - l_0)\; - k(z(t) - z_0) = m\ddot{z}(t)$$

There remains:

$$\ddot{z}\,(t) + \frac{k}{m}\,(z(t) - z_0) = 0 \tag{9.10}$$

Let $Z = z - z_0$. Equation (9.10) becomes:

$$\ddot{Z} + \frac{k}{m}Z = 0 \tag{9.11}$$

Z denotes the deviation from the equilibrium position.

9.1.2.3. Mechanical energy

The two forces $\vec{P}$ and $\vec{T}$ involved are conservative. $\vec{P}$ and $\vec{T}$ are carried by the axis (Oz):

$$\overrightarrow{rot}\vec{P} = \overrightarrow{rot}\vec{T} = \vec{0} \tag{9.12}$$

Where from

$$\vec{P} = -\overrightarrow{grad}E_{p1} \qquad \text{et} \qquad \vec{T} = -\overrightarrow{grad}E_{p2} \tag{9.13}$$

The potential energies E_{p1} and E_{p2} from which forces are derived $\vec{P}$ and $\vec{T}$ are respectively:

$$\begin{cases} E_{p1} = -mgz + A_1 \\ E_{p2} = k\left(\frac{z^2}{2} - l_0 z + A_2\right) = \frac{k}{2}(z - l_0)^2 + A_3 \end{cases} \tag{9.14}$$

A_1 , A_2 and A_3 are integration constants. The potential energy of the system is:

$$E_p = E_{p1} + E_{p2} = -mgz + \frac{k}{2}(z - l_0)^2 + A_4 \tag{9.15}$$

With

$$A_4 = A_1 + A_3 \tag{9.16}$$

To determine the constant A_4, we take the potential energy E_p to be zero at equilibrium ($E_p(z_0) = 0$), which gives:

$$A_4 = mgz_0 - \frac{k}{2}(z - l_0)^2 \tag{9.17}$$

Where from

$$E_p = \frac{k}{2}(z - z_0)^2 = \frac{1}{2}kZ^2 \tag{9.18}$$

The kinetic energy of the system is:

$$E_c = \frac{1}{2}mV^2 = \frac{1}{2}m\dot{Z}^2 \tag{9.19}$$

The mass-spring system is a conservative system. Its mechanical energy remains constant. Therefore:

$$E_m = E_c + E_p = \frac{1}{2}m\dot{Z}^2 + \frac{1}{2}k\dot{Z}^2 = Cte \tag{9.20}$$

Where from

$$\frac{dE_m}{dt} = m\dot{Z}\ddot{Z} + kZ\dot{Z} = 0 \tag{9.21}$$

And

$$\ddot{Z} + \frac{k}{m}Z = 0 \tag{9.22}$$

which can also be written as,

$$\ddot{Z} + \omega_0^2 Z = 0 \tag{9.23}$$

The frequency of the free oscillator is:

$$\omega_0^2 = \frac{k}{m} \tag{9.24}$$

The solution to equation (9.23) is of the form:

$$Z(t) = A\sin(\omega_0 t + \varphi) \tag{9.25}$$

Therefore, the mechanical energy of the mass-spring system is written as:

$$E_m = \frac{1}{2}kA^2 \tag{9.26}$$

This energy is proportional to the square of the amplitude of the oscillations. The period of the oscillations is independent of the amplitude *A* and is written as:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{9.27}$$

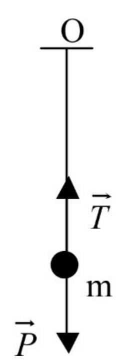
9.1.3 Simple pendulum

Consider an inextensible string of negligible mass compared to m. The mass is attached to one end of the string, the other end is fixed at a point O.

9.1.3.1. Pendulum at equilibrium

At equilibrium, the vector sum of the weight $\vec{P}$ of the mass *m* and the tension $\vec{T}$ of the string is zero:

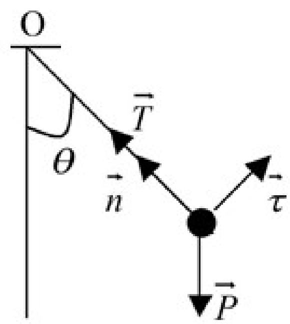
$$\vec{P} + \vec{T} = \vec{0} \tag{9.28}$$



9.1.3.2. Pendulum out of equilibrium

The fundamental principle of dynamics is written as:

$$\vec{P} + \vec{T} = m\vec{\gamma} \tag{9.29}$$



Projecting this vector equation onto the axes of the Serret-Frenet trihedron gives the following two scalar equations:

$$\begin{cases} -mg\sin\theta = m\gamma_t = m\dfrac{dV}{dt} = m\dfrac{d^2s}{dt^2} \\ T - mg\cos\theta = m\gamma_n = m\dfrac{V^2}{l} \end{cases} \tag{9.30}$$

Where s is the curvilinear abscissa of the motion. l is the length of the string = radius of curvature of the particle's trajectory.

In the case of small oscillations, sin θ close to θ, equation (9.30) gives:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{9.31}$$

This differential equation has the following solution:

$$\theta(t) = \sin(\omega_0 t + \varphi) \tag{9.32}$$

where

$$\omega_0 = \sqrt{\frac{g}{l}} \quad \text{et} \quad T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}} \tag{9.33}$$

T_0 denotes the period of the oscillations.

9.2 Oscillators damped by fluid friction

In this section, we take into account the friction forces between the mass and the fluid.

There are two types of friction:

- Solid friction, where the friction force is a constant.
- Fluid (or viscous) friction, where the friction force is proportional to the velocity vector of the mass m.

$$\vec{F}_f = -k'\vec{V} \tag{9.34}$$

k' is the coefficient of friction. k' is positive. The sign (−) that appears in the expression for the friction force reflects the fact that this force opposes motion. In the one-dimensional case, we have:

$$\vec{V} = \dot{z}\vec{k} \tag{9.35}$$

The fundamental principle of dynamics applied to the mass-spring system damped by fluid friction is:

$$m\ddot{z} = -kz - k'\dot{z} \Longrightarrow \ddot{z} + \frac{k'}{m}\dot{z} + \frac{k}{m}z = 0 \tag{9.36}$$

This is a second-order linear differential equation with constant coefficients and no second member. We set $\frac{k}{m} = \omega_0^2$, with ω_0= natural frequency of the oscillator, and $\frac{k'}{m} = 2\lambda\omega_0$, where λ is the damping coefficient. The differential equation of motion of point *M* then becomes:

$$\ddot{z} + 2\lambda\omega_0\dot{z} + \omega_0^2 z = 0 \tag{9.37}$$

whose characteristic equation is:

$$r^2 + 2\lambda\omega_0 r + \omega_0^2 = 0 \tag{9.38}$$

The discriminant of this equation is:

$$\Delta' = \omega_0^2(\lambda^2 - 1) \tag{9.39}$$

We therefore have three cases to distinguish:

a) Δ*'* > **0** (or ***λ*** > **1**)**:**

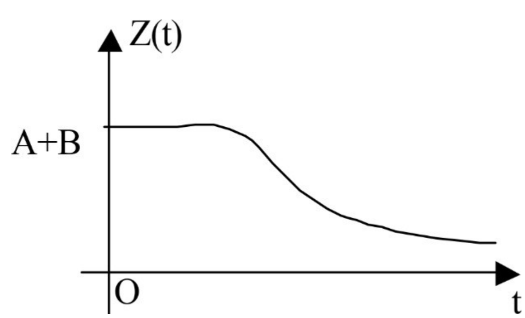
The two roots of the above characteristic equation are:

$$\begin{cases} r_1 = -\lambda\omega_0 + \omega_0\sqrt{\lambda^2 - 1} \\ r_2 = -\lambda\omega_0 - \omega_0\sqrt{\lambda^2 - 1} \end{cases} \tag{9.40}$$

The solution to the equation of motion of point *M* is therefore

$$z(t) = e^{-\lambda\omega_0 t}\left[Ae^{\omega_0 t\sqrt{\lambda^2-1}} + Be^{-\omega_0 t\sqrt{\lambda^2-1}}\right] \tag{9.41}$$

A and *B* are constants to be determined from the initial conditions. When $t \to \infty$, $z(t) \to 0$. In this case, there are no oscillations around the equilibrium position. There is a return to equilibrium after a sufficiently long time. The regime is *aperiodic*.



b) $\Delta' = 0$ **(or $\lambda = 1$):**

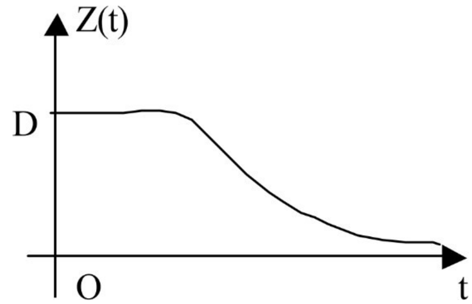
The roots of the characteristic equation are:

$$r_1 = r_2 = r = -\omega_0 \tag{9.42}$$

The solution to the differential equation of motion is:

$$z(t) = e^{-\omega_0 t}[Ct + D] \tag{9.43}$$

In this case, the return to equilibrium occurs more quickly than in the aperiodic regime. This is the *aperiodic- critical* regime.



c) $\Delta' < 0$ **(or $0 < \lambda < 1$):**

The roots of the characteristic equation are:

$$\begin{cases} r_1 = -\lambda\omega_0 + i\omega_0\sqrt{\lambda^2 - 1} = -\lambda\omega_0 + i\omega \\ r_2 = -\lambda\omega_0 - i\omega_0\sqrt{\lambda^2 - 1} = -\lambda\omega_0 - i\omega \end{cases} \tag{9.44}$$

where i is the complex number ($i^2 = -1$) and $\omega = \omega_0\sqrt{\lambda^2 - 1}$ denotes the pseudo-period of the oscillator under study. The solution $z(t)$ can then be written as:

$$z(t) = e^{-\lambda\omega_0 t}\left[C_1 e^{i\omega t} + C_2 e^{-i\omega t}\right] \tag{9.45}$$

C_1 and C_2 are constants that will be determined from the initial conditions. This solution can also be written as

$$z(t) = e^{-\lambda\omega_0 t} A_1 \sin(\omega t + \varphi) \tag{9.46}$$

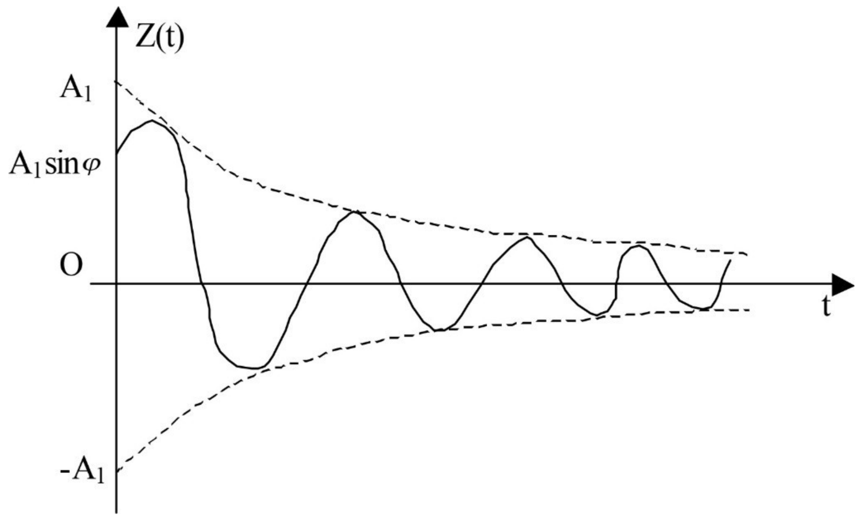
$A_1 e^{-\lambda\omega_0 t}$ and φ are respectively the amplitude and phase of the oscillation. The regime is *pseudo-periodic*. The pseudo-period of the oscillations is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}\frac{1}{\sqrt{1 - \lambda^2}} \tag{9.47}$$

Or,

$$T = \frac{T_0}{\sqrt{1 - \lambda^2}} \tag{9.48}$$

with $T_0 = \frac{2\pi}{\omega_0}$ is the natural frequency of the oscillator. The pseudo-period is therefore greater than the natural period of the oscillator ($T > T_0$).



Logarithmic decrement:

We have:

$$z(t) = e^{-\lambda\omega_0 t} A_1 \sin(\omega t + \varphi) \tag{9.49}$$

and

$$z(t+T) = e^{-\lambda\omega_0 (t+T)} A_1 \sin(\omega t + \varphi) \tag{9.50}$$

We define the logarithmic decrement δ by the following ratio:

$$e^{-\delta} = \frac{z(t+T)}{z(t)} = e^{-\lambda\omega_0 T} \tag{9.51}$$

Therefore

$$\delta = \lambda\omega_0 T = \ln\left(\frac{z(t+T)}{z(t)}\right) \tag{9.52}$$

The logarithmic decrement characterizes the decrease in elongation at each period.

Note:

The logarithmic decrement can also be written as

$$\delta = \lambda\omega_0 T = \lambda\omega_0 \frac{2\pi}{\omega_0}\frac{1}{\sqrt{1-\lambda^2}} = \frac{2\pi\lambda}{\sqrt{1-\lambda^2}} \tag{9.53}$$

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