Exercise 1 (5 marks)

1) We solve the linear equation

$$y' - \frac{1}{x}y = -\ln x, \qquad (x > 0)$$

This is a linear differential equation of the form

$$y' + P(x)y = Q(x) \quad with P(x) = -\frac{1}{x}, \quad Q(x) = -\ln x.$$

a) The integrating factor is

$$\mu(x) = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}.$$

b) **The general solution is :**

$$y = \frac{1}{\mu(x)}\int \mu(x)Q(x)dx = x\int \frac{-1}{x}lnx\,dx$$

Therefore : $y = x\left(C - \frac{(\ln x)^2}{2}\right), c \in \mathbb{R}$

2) We deduce the solution of Bernoulli equation

: $z' + \frac{1}{x}z = (\ln x)z^2$
This is a Bernoulli equation :

$$z' + P(x)z = Q(x)z^n, \quad (with \quad (P(x) = \frac{1}{x}, \quad Q(x) = \ln x, n = 2).$$

We put

$$u = z^{1-n} = z^{-1} = \frac{1}{z}.$$

Then

$$u' = -\frac{z'}{z^2}$$

Multiply by $-z^{-2}$:

$$u' - \frac{1}{x}u = -\ln x.$$

This is exactly the previous equation, since $u = \frac{1}{z}$, hence : $z(x) = \frac{1}{x\left(C - \frac{(\ln x)^2}{2}\right)}$

Exercise 2 (7 marks)

Consider : $y'' - 3y' + \lambda y = e^x$.

1) Values of λ for two real roots

The characteristic equation is

$$r^2 - 3r + \lambda = 0. \quad The\ discriminant\ is: \ \Delta = 9 - 4\lambda.$$

For two real roots :$\Delta \geq 0$, which gives $\lambda \leq \frac{9}{4}$

2)We solve

$y'' - 3y' + 2y = 0.$
Characteristic equation : $r^2 - 3r + 2 = 0$, the discriminant is :$\delta = 1$.
Hence the roots are

$$r_1 = 1, \qquad r_2 = 2.$$

Therefore the general solution is

$$y_h(x) = C_1 e^x + C_2 e^{2x}$$

3) a) Particular solution of : $y'' - 3y' + 2y = e^x$

Since : $\lambda = 1$ e^x is simple root of characteristic-equ ,then

$$y_p = Axe^x.$$

Compute derivatives :

$$y_p' = A(x+1)e^x,$$

$$y_p'' = A(x+2)e^x.$$

Substitute :

$$A(x+2)e^x - 3A(x+1)e^x + 2Axe^x = e^x.$$

Simplify :

$$A[-1]e^x = e^x.$$

Thus

$$A = -1.$$

Hence

$$y_p = -xe^x$$

3b) General solution of (E_2)

The general solution is

$$y = y_h + y_p.$$

Therefore

$$y(x) = C_1 e^x + C_2 e^{2x} - xe^x$$

Exercise 3 (8 marks)

Consider : $f(x,y)=\frac{x\sin x}{x^2+y^2}$

1) Definition domain

The denominator must be nonzero : $x^2+y^2\neq 0$
Hence

$$D_f=\mathbb{R}^2\setminus\{(0,0)\}$$

2) Partial derivatives

Partial derivative with respect to x

$$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)(\sin x+x\cos x)-2x(x\sin x)}{(x^2+y^2)^2}.$$

Thus

$$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)(\sin x+x\cos x)-2x^2\sin x}{(x^2+y^2)^2}$$

Partial derivative with respect to y

$$\frac{\partial f}{\partial y}=\frac{0-(x\sin x)(2y)}{(x^2+y^2)^2}.$$

Hence

$$\frac{\partial f}{\partial y}=-\frac{2xy\sin x}{(x^2+y^2)^2}$$

Values at $(1,0)$

- For : $\frac{\partial f}{\partial x}(1,0)=\frac{1(sin1+cos1)-2sin1}{1}=cos1-sin1$
- • For : $\frac{\partial f}{\partial y}(1,0)$=0 :

3) A) Continuity of g :

$$g(x,y)=\begin{cases}\frac{x.sinx}{x^2+y^2}, & if(x,y)\neq(0,0)\\ 0, & if(x,y)=(0,0)\end{cases}$$

a) **We study continuity at** $(0,0)$**.**
Take $x=y$:

$$g(x,x)=\frac{x\sin x}{2x^2}=\frac{\sin x}{2x}.$$

Then : $\lim_{x\to 0}\frac{\sin x}{2x}=\frac{1}{2}$
Hence

$$\lim_{(x,y)\to(0,0)}g(x,y)\neq g(0,0).$$

Therefore g **is not continuous** at $(0,0)$. Or : $\lim_{x\to 0}g(x,0)=1\neq g'0,0)$ **b) On** $\mathbb{R}^{*2}$ **:** The function g is continuous because it is a fraction of two functions continuous.

3 B) Differentiability

Since g **is not continuous at** $(0,0)$, it follows that : g **is not differentiable at (0,0)**