People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
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Department of Basic Education in Science and Technology
1^{st} year

Duration : 2 Hours.

Correction of Exam : Algebra 2

Exercise 1. *(Matrix+Linear equation system 10pts)*

Let the matrix A defined by : $A = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$

1. *Calculate :* $\det(A)$, $Tr(A)$.
2. *Determine the matrices B and C such that :* $B = A + 2A^t$, $C = A^2$.
3. *Show that : A is invertible and calculate A^{-1}.*
4. *Give the linear application f associated with the matrix A.*
5. *Is f bijective? If yes, give f^{-1}.*
6. *Let the system :* $S = \begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases}$
   - *Write (S) in matrix form.*
   - *Deduce the solution of the system (S) using the matrix method.*

Correction

Let the matrix A defined by : $A = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix}$

1. Calculate : $\det(A)$, $Tr(A)$.

$$\det(A) = 4\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} -5 & -1 \\ 2 & 1 \end{vmatrix} + 1\begin{vmatrix} -5 & -2 \\ 2 & 1 \end{vmatrix} = 1$$

$$Tr(A) = 4 + (-2) + 1 = 3$$

2. Determine the matrices B and C such that : $B = A + 2A^t$, $C = A^2$.

$$B = A + 2A^t = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} + 2\begin{pmatrix} 4 & -5 & 2 \\ 2 & -2 & 1 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 12 & -8 & 5 \\ -1 & -6 & 1 \\ 4 & -1 & 3 \end{pmatrix}$$

$$C = A^2 = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 8 & 5 & 3 \\ -12 & -7 & -4 \\ 5 & 3 & 2 \end{pmatrix}$$

3. Show that : A is invertible and calculate A^{-1}.

   We have : $\det(A) = 1 \neq 0$, then : A is invertible.

   To find A^{-1}, we use the formula :

$$A^{-1} = \frac{1}{\det(A)} \cdot com(A)^t$$

   We have :

$$Com(A) = \begin{pmatrix} -1 & 3 & -1 \\ -1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

   then :

$$A^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & & 2 \end{pmatrix}$$

4. Give the linear application f associated with the matrix A :

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

   $\forall X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3,$

$$f(x, y, z) = A \cdot X = \begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (4x + 2y + z, -5x - 2y - z, 2x + y + z).$$

5. Is f bijective ? If yes, give f^{-1}.

   We have : A is invertible ($\det(A) = 1 \neq 0$), then : f is bijective.

   With :

$$f^{-1}(x, y, z) = A^{-1} \cdot X = (-x - y, 3x + 2y - z, -x + 2z)$$

6. Let the system : $S = \begin{cases} 4x + 2y + z = 1 \\ -5x - 2y - z = 2 \\ 2x + y + z = -1 \end{cases}$

   — Write (S) in matrix form :

$$\begin{pmatrix} 4 & 2 & 1 \\ -5 & -2 & -1 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

— Deduce the solution of the system (S) using the matrix method :

We have :

$$A \cdot X = B \Leftrightarrow X = A^{-1} \cdot B$$

$$X = \begin{pmatrix} -1 & -1 & 0 \\ 3 & 2 & -1 \\ -1 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ -3 \end{pmatrix}$$

- The solution is : $S = \{(-3, 8, -3)\}$

Exercise 2. *(Vector Spaces 5pts) Consider the subset F of $\mathbb{R}^3$ defined by :*

$$F = \{(x, y, z) \in \mathbb{R}^3 \backslash x - 2y + 3z = 0\}$$

1. *Show that F is a vector subspace of $\mathbb{R}^3$.*
2. *Find a basis of F and determine its dimension.*
3. *Let the vector : $v = (1, 1, 1)$. Is $v \in F$?*
4. *Is $F = \mathbb{R}^3$?*

Correction

1. F is a subspace of $\mathbb{R}^3$.

1) $F \neq \varnothing \Leftrightarrow 0_{\mathbb{R}^3} = (0, 0, 0) \in F$ because : $0 - 2(0) + 3(0) = 0$ is true.

2) Let : $\begin{cases} X = (x, y, z) \in F \\ Y = (x', y', z') \in F \end{cases} \Longleftrightarrow \begin{cases} x - 2y + 3z = 0 \\ x' - 2y' + 3z' = 0 \end{cases}$

$$X + Y = (x + x', y + y', z + z') \in F?$$

$$(x + x') - 2(y + y') + 3(z + z') = 0?$$

We have : $x + x' - 2y - 2y' + 3z + 3z' = x - 2y + 3z + x' - 2y' + 3z' = 0 + 0 = 0$

Then : $X + Y \in F$

3) Let : $\begin{cases} X = (x, y, z) \in F \\ \lambda \in \mathbb{R} \end{cases} \Longleftrightarrow \begin{cases} x - 2y + 3z = 0 \\ \lambda \in \mathbb{R} \end{cases}$

$$\lambda \cdot X = \lambda \cdot (x, y, z) = (\lambda x, \lambda y, \lambda z) \in F?$$

$$\lambda x - 2\lambda y + 3\lambda z = 0??$$

We have : $\lambda x - 2\lambda y + 3\lambda z = \lambda(x - 2y + 3z) = \lambda(0) = 0$

Then : $\lambda \cdot X \in F$.

From 1), 2) and 3) we deduce that : F is vector subspace of $\mathbb{R}^3$.

2. **Determine a basis of F.**

• Let : $u = (x, y, z) \in F$ with : $x - 2y + 3z = 0$.

$$x - 2y + 3z = 0 \Longrightarrow x = 2y - 3z$$

We have :

$$\begin{aligned} u = (x, y, z) &= (2y - 3z, y, z) \\ &= y(2, 1, 0) + z(-3, 0, 1). \end{aligned}$$

Then : $F = \text{Vect}((2, 1, 0), (-3, 0, 1)) = \text{Vect}(v_1, v_2)$.

So : $\{v_1, v_2\}$ is a generating family of F. (1)

$\{v_1, v_2\}$ is a linearly independent of F?

• Let : $\alpha, \beta \in \mathbb{R}$.

$$\alpha v_1 + \beta v_2 = 0_{\mathbb{R}^3} \Leftrightarrow \alpha(2, 1, 0) + \beta(-3, 0, 1) = (0, 0, 0)$$

$$\begin{cases} 2\alpha - 3\beta = 0 \\ \alpha = 0 \\ \beta = 0 \end{cases}$$

$$\begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}$$

Then : $\{v_1, v_2\}$ is a linearly independent of F. (2)

From (1) and (2), we deduce that : $\{v_1, v_2\}$ is a basis of F and $\dim F = 2$.

3. Let the vector : $v = (1, 1, 1)$. Is $v \in F$?

We check if v satisfy the equation $x - 2y + 3z = 0$:

$$1 - 2(1) + 3(1) = 1 - 2 + 3 = 2$$

Since : $2 \neq 0$, the vector : $v \notin F$.

4. $F \neq \mathbb{R}^3$ because : $\dim F = 2 \neq 3 = \dim \mathbb{R}^3$.

Exercise 3. *(Linear Applications 5pts) Consider the following application :*

$$\begin{aligned} f : \mathbb{R}^3 &\longrightarrow \mathbb{R}^2 \\ (x, y, z) &\longrightarrow f(x, y, z) = (2x - y,\ y + 3z). \end{aligned}$$

1. *Show that f is a linear application.*
2. *Determine* $\ker(f)$. *Is f injective? Justify your answer.*
3. *Determine $Im(f)$ and deduce* $\dim Im(f)$. *Is f surjective? Justify your answer.*

4. *Is f bijective ?*

Correction

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ defined by :

$$f(x, y, z) = (2x - y,\ y + 3z), \quad for\ all\ (x, y, z) \in \mathbb{R}^3$$

1. Show that f is a linear application.

   - Let : $X = (x, y, z)$ and $Y = (x', y', z') \in \mathbb{R}^3$, we have :

$$\begin{aligned} f(X+Y) &= f(x+x', y+y', z+z') \\ &= (2(x+x') - (y+y'), (y+y') + 3(z+z')) \\ &= ((2x-y) + (2x'-y'), (y+3z) + (y'+3z')) \\ &= (2x-y, y+3z) + (2x'-y', y'+3z') \\ &= f(x,y) + f(x',y') \\ &= f(X) + f(Y). \end{aligned}$$

   Then : $f(X+Y) = f(X) + f(Y)$..... (1)

   - Let : $\lambda \in \mathbb{R}$ and $X = (x, y, z) \in \mathbb{R}^3$.

$$\begin{aligned} f(\lambda X) &= f(\lambda x, \lambda y, \lambda z) \\ &= (2\lambda x - \lambda y, \lambda y + 3\lambda z) \\ &= \lambda(2x - y, y + 3z) \\ &= \lambda f(x, y, z) \\ &= \lambda f(X). \end{aligned}$$

   Then : $f(\lambda X) = \lambda f(X)$..... (2)

   From (1) and (2), we conclude that : f is a linear application.

2. Determine $\ker(f)$. Is f injective ? Justify your answer.

   - **From the kernel**

$$\begin{aligned} \ker(f) &= \{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) = 0_{\mathbb{R}^2}\} \\ &= \{(x, y, z) \in \mathbb{R}^3 \mid (2x - y, y + 3z) = (0, 0)\} \\ &= \{(x, y) \in \mathbb{R}^2 \mid 2x - y = 0 \text{ and } y + 3z = 0\} \end{aligned}$$

   We solve the system : $\begin{cases} 2x - y = 0 \\ y + 3z = 0 \end{cases} \Rightarrow \begin{cases} y = 2x \\ z = -\frac{2}{3}x \end{cases}$

$$\begin{aligned} \ker(f) &= \{(x, 2x, -\frac{2}{3}x) \mid x \in \mathbb{R}\} \\ &= \{x(1, 2, -\frac{2}{3}) \mid x \in \mathbb{R}\} \\ &= \{(1, 2, -\frac{2}{3})\} \end{aligned}$$

- We have : $\ker(f) = \{(1, 2, -\frac{2}{3})\}$, then : f is not injective.

3. Determine $Im(f)$ and deduce $\dim Im(f)$. Is f surjective ? Justify your answer.

- **From the image :** We can find the image by applying f to the canonical basis of $\mathbb{R}^3$:

$$f(1,0,0) = (2,0)$$
$$f(0,1,0) = (-1,1)$$
$$f(0,0,1) = (0,3)$$
$$Im(f) = \text{Vect}((2,0),(-1,1),(0,3))$$

$\dim(Im(f)) = 2$. (because : $\dim(Im(f)) = \dim(\mathbb{R}^3) - \dim\ker(f) = 3 - 1 = 2$)

We have : $\dim(Im(f)) = 2 = \dim \mathbb{R}^2$ $(Im(f) = \mathbb{R}^2)$, Then : f is surjective. (2)

From (1) and (2), we deduce that : f is not bijective.