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Ministry of Higher Education and Scientific Research

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Faculty of Sciences and Technology (ST) EBST Department

MECHANICS OF THE MATERIAL POINT

(Course and solved exercises)

Matter: Physics1 (Mechanics)

Specialty: Common trunk science and technology

Level: License (L1) 1st year S1

Established By:

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What is Mechanics?



Mechanics studies the motion of objects and the forces that cause them.

It explains why things move, stop, or stay still – using Newton's Laws.















Mechanics of material point

Course Objectives

This course aims to provide the introduce basic concepts of material point mechanics in physics, this handout is made up of four chapters consistent with the programs for the first semester.is aimed at students in the core areas (first year) of science and technology (ST), material sciences (SM).

• ***Chapter I***: We will start by presenting a mathematical reminders, where in the first part we will we will study dimensional analysis while in the second part, we will focus on the vector calculation, both cover the fundamental quantities that are used to express physical laws and principles. In addition to necessary reviews, this section aims to introduce clear definitions and appropriate notations.

• ***Chapter II***: is dedicated to the kinematics of the material point. The objective of this chapter is to describe the movement of the material point (position, velocity, acceleration) in the different coordinate systems.

• ***Chapter III***: dynamics of the material point within the framework of Newton's mechanics laws, including its three laws or principles: the law of inertia, the fundamental law of dynamics, and the law of reciprocal actions. We consider general laws, referred to as force laws, established for several interactions along with applications aimed are predicting the motion of bodies.

• ***Chapter IV*****:** Work, Power and Energy. We first consider the work done by a force and the kinetic energy of a particle. Subsequently, we delve into the concepts of potential and total energies, and we apply the principle of conservation of energy in various practical situations, to solve complex problems where force, distance, and time are related.

To achieve a correct understanding of the lessons, we have included with each chapter a set of exercises with the typical and detailed solution.

General objectives:

- Identify the concept of vector calculus and its properties.
- Learn to use the usual coordinate systems
- Interpret the movement of a point over time independently of the causes, which produce this movement.
- Introduce the student to the basics of Newtonian physics through Kinematic

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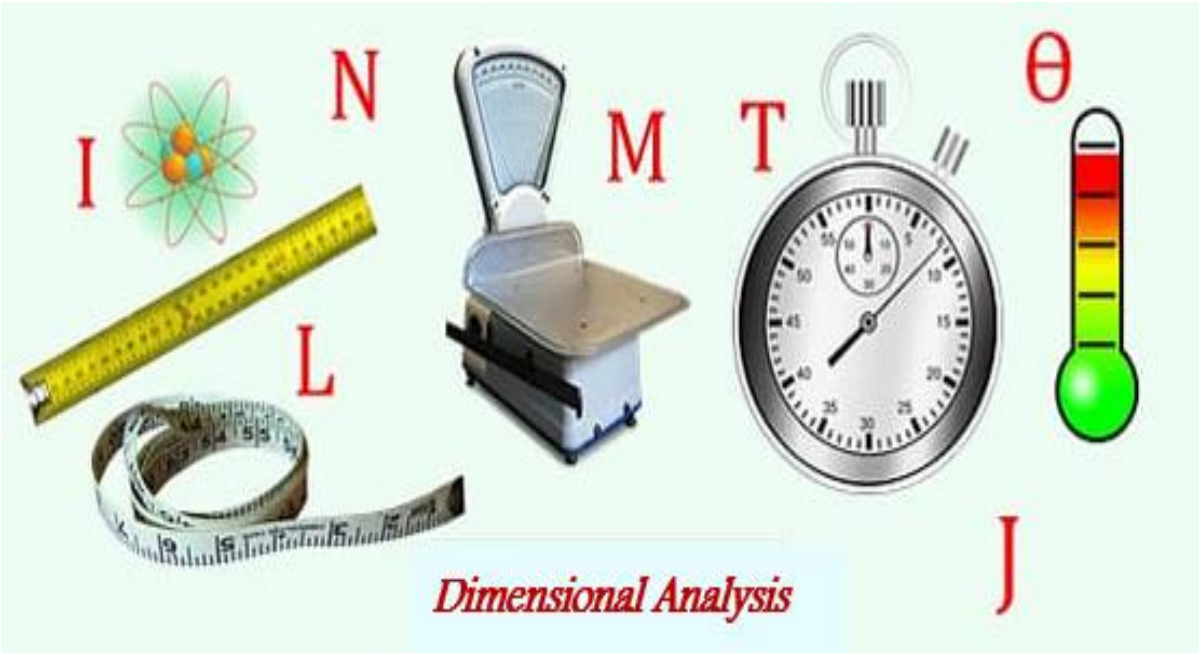
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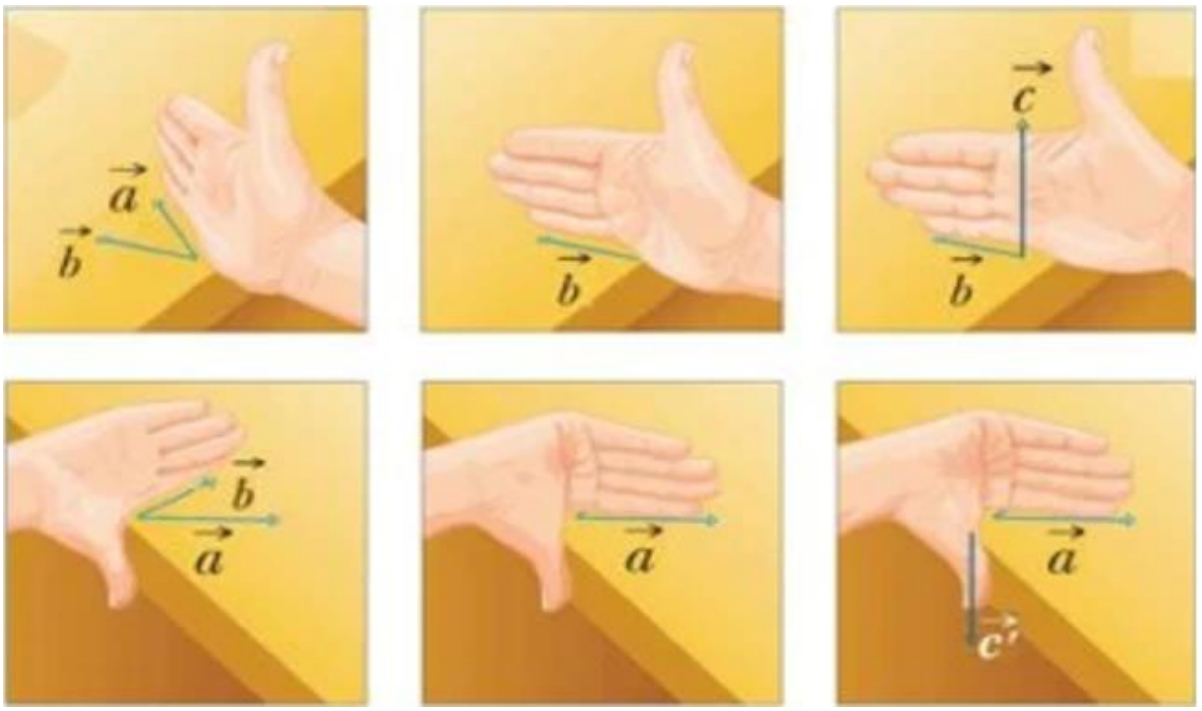
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CHAPTER I: MATHEMATICAL TOOLS





Chapter I: Mathematical tools

Part 1: Dimensional Analysis

I.1. Dimensional Analysis:

Dimensional analysis involves verifying the relationships between physical quantities by identifying their dimensions. These dimensions are independent of numerical multiples and constants, and all quantities in the universe can be expressed in terms of fundamental dimensions.

I.1.1. Physical Quantity:

A physical quantity is any quantity that can measured or calculated from other quantities called base quantities. For example, distance and time can be define by measuring both, while speed is calculates by dividing the distance traveled by the travel time. In this case, distance and time are base quantities, and speed is a derived quantity. Most physical quantities are derive from one another and from a few base quantities.

Examples of physical quantities: mass, time, distance, area, volume, energy, density, pressure...velocity, acceleration, force and temperature …. Etc.

I.1.2. International System of Units:

The value of a physical quantity is given in relation to a standard known as a "unit". The first four fundamental units constitute the MKSA International System (Meter, Kilogram, Second, Ampere). Using these fundamental units, derived units can be constructed: area (m^2), velocity (m/s), force (kg m/s^2)…

The International System of Units (SI) consists of seven base (fundamental) units corresponding to seven physical quantities, as summarized in the following table:

The Fundamental Quantities	*Symbols*	*Dimensions*	*Units (in the international system (SI))*
Length	*l,x,r*	*L*	*Meter* ***(m)***
Mass	*m*	*M*	*Kilogram* ***(kg)***
Time	*t*	*T*	*Second* ***(s)***
Current intensity	*i*	*I*	*Ampere* ***(A)***
Temperature	*T*	*θ*	*Kelvin* ***(K)***
Light intensity	*j*	*J*	*Candela* ***(Cd)***
Quantity of material	*n*	*N*	*Mole* ***(mol)***

Note: There are also other systems of units in physics, such as:

- The International System (SI) known as MKSA (Meter, Kilogram, Second, Ampere), which is the most widely used system.

- The CGS system (Centimeter, Gram, Second), which is less commonly used.

I.1.3. Definition of dimensions:

In physics, the term "dimension" signifies the inherent property of a physical quantity. We use dimension to classify the physical quantities according to their types regardless of their values or the unit used for them. For instance, the measurement of distance between two points can be done using units like feet, meters, or kilometers', with each unit representing a unique aspect of the dimension known as length. Similarly, when measuring the mass of an object, it can be quantified using units like kilogram, gram, pound, or milligram. These units all share a common nature, as they represent mass, and therefore they have a mass dimension. To denote the dimension, we typically write the symbol of the quantity within square brackets **[quantity],** dim G = [G]

I.1.3.1. Dimensional Equations:

Or dimensional equations. We call the equation with dimensions for the amount G the equation that takes the form $[G] = M^{\alpha}L^{\beta}T^{\gamma}$

By denoting M, L, and T as the dimensions of the fundamental quantities mass, length, and time, we can express the dimensions of other derived quantities in terms of these three fundamental dimensions.The resulting equations are the dimension equations for these physical quantities.

Quantity	symbol	Unit	Dimension
Velocity	v	ms^{-1}	L^1T^{-1}
Acceleration	a	ms^{-2}	L^1T^{-2}
Frequency	f	HZ	T^{-1}
Volume	V	m^3	L^3
Force	F	N	$M^1L^1T^{-2}$
Work	W	J	$M^1L^2T^{-2}$
Power	P	w	$M^1L^2T^{-3}$
Pressure	p	Pa	$M^1L^{-1}T^{-2}$
Energy	W, E	$N.m$	$M^1L^2T^{-2}$
Electric potential	V	V	$M^1L^2T^{-3}I^{-1}$
Electric resistance	R	Ω	$M^1L^2T^{-3}I^{-2}$
Electric field	E	V/m or N/C	$M^1L^1T^{-3}I^{-1}$
Electric charge	q	C	I^1T^1

Example 1: we can derive a the this relation of the acceleration. The relation reads

$$[a] = \frac{[speed]}{[time]} = \frac{[v]}{[t]} = \frac{LT^{-1}}{T} = LT^{-2}$$

the unit of acceleration is (m/s^2).

Notes:

* Dimensionless quantities: all exponents are zero.

* Exponential, logarithmic, and trigonometric functions, as well as constants and everything within these functions, have a dimension of one.

$[e^x] = 1$, [Log x] = 1, [4] = 1, [sin x] = 1, [π] = 1...

I.1.3.2. Dimension of a Derived Quantity:

To calculate the dimension of a derived (non-fundamental) physical quantity, we need an equation that relates it to the fundamental quantities. This is also possible using the unit of the quantity. In the table below, we give some examples of derived quantities and their dimensions.

Quantity	Equation	Dimension
Angular speed, pulsation: ω	*α/t*	T^{-1}
Angular acceleration:ω̇	*w/t*	T^{-2}
Frequency: f	*1/T*	T^{-1}
Force: F	*m.a*	$M.L.T^{-2}$
Moment of inertia: J	$m.l^2$	$M.L^2$
Pressure: Pr	*F/S*	$M.L^{-1}.T^{-2}$
Work: W	*F.d*	$M.L^2.T^{-2}$
Power: P	*W/t*	$M.L^2.T^{-3}$
Charge: Q	*i.t*	*I.T*
Magnetic field/F	*F/q.V*	$M.T^{-2}.I^{-1}$
Electric field: E	*V/l*	$M.L.T^{-3}.I^{-1}$
Capacity : Q	*Q/V*	$M^{-1}.L^{-2}.T^4.I^2$
Resistance: R	*V/i*	$M.L^2.T^{-3}.I^{-2}$

Part 2: Vector Analysis

I.2. Vector Analysis:

The use of vectors in mechanics is fundamental. They are used to represent velocities and accelerations of points, rotations of solids, displacements and exerted forces.

I.2.1. Scalar quantities and vector quantities:

The physical quantities that used to describe the motion of objects can divided into two categories. The quantity is either a vector or a scalar. These two categories can be distinguishes from one another by their distinct definitions:

Scalar quantity: are quantities that are fully described by a magnitude (or numerical value) alone. The word scalar means representable by position on a line; having only magnitude. (Example: distance, area, volume, speed, mass, time, energy, density, temperature, and work).

Vector quantity: that are fully describes by both a magnitude and a direction. (Example: velocity, force and acceleration).

A vector is a line segment AB, having an origin A and an end B. We denote it by $\overrightarrow{AB}$, characterized by:

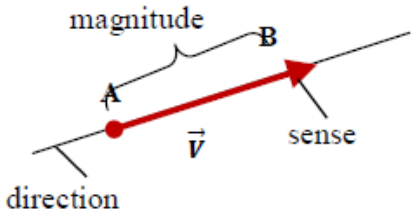
- Its direction, which is define by that of the line, which carries the segment

- Its sense, which designates the orientation of the vector (from A towards B).

- Its magnitude (norm or intensity) which is equal to the length of the segment [AB] noted $\overrightarrow{AB}$ which is always positive.

Note:

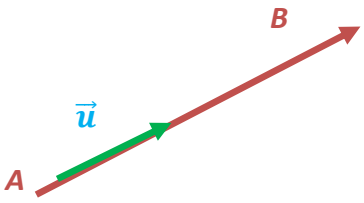
A single letter can designate a vector $\overrightarrow{AB} = \vec{V}$.



I.2.2.Unit vector:

The unit vector is a vector whose magnitude is equal to one. We write $|\vec{u}| = 1$ parallel to the unit vector $\vec{\mu}$ such that $\overrightarrow{AB} = AB.\vec{\mu}$ with: $\left\|\overrightarrow{AB}\right\| = AB$

$$\vec{\mu} = \frac{\overrightarrow{AB}}{\left\|\overrightarrow{AB}\right\|}$$



I.2.3. Components of a vector:

A vector is described by its components, which are determined from a reference frame.

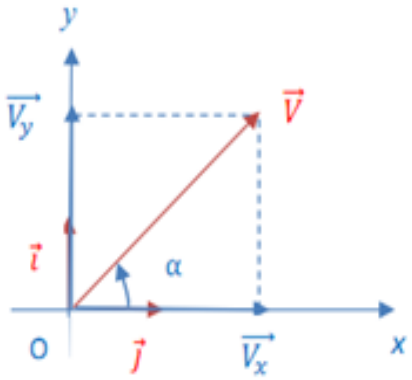
This reference frame can be linear (a single x component), planar (two components x,y) or in space (three components x,y and z).

I.2.3.1. Vector coordinates in the Cartesian reference frame:

The Cartesian reference frame is an orthonormal frame: unit vectors must be mutually orthogonal to each other and normalized to unity.

❖ ***In the plane*** $(o,\vec{\imath},\vec{\jmath})$

$$\begin{cases} \vec{V} = \vec{V}_x + \vec{V}_y \\ \vec{V}_x = V_x\vec{\imath} \\ \vec{V}_y = V_y\vec{\jmath} \\ V_x = V\cos\alpha\,;\, V_y = V\sin\alpha \end{cases} \Rightarrow \vec{V} = V_x\vec{\imath} + V_y\vec{\jmath}$$



The modulus of vector $(\vec{V})$ *is calculated by :*

$$V = \sqrt{V_x^2 + V_y^2}$$

The components of vector $(\vec{V})$ in the orthonormal plane $(O,\vec{\imath},\vec{\jmath})$ are: (V_x) and (V_y) and we can writ

$$\vec{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix}$$

$$\vec{V} = V(\cos\alpha\vec{\imath} + \sin\alpha\vec{\jmath})$$

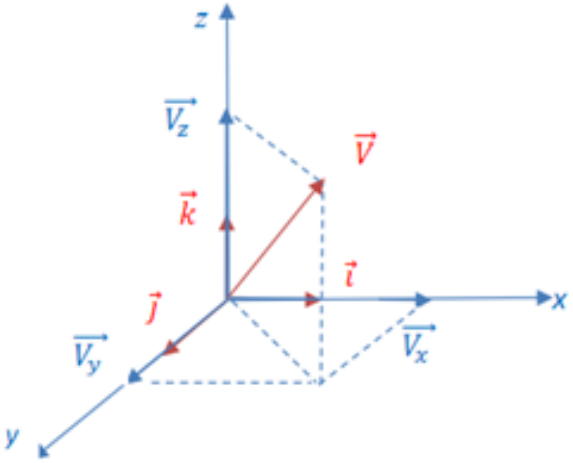
❖ ***In the space*** $(O,\vec{\imath},\vec{\jmath},\vec{k})$

The coordinates of a vector in space, represented in an orthonormal base frame R $(O,\vec{\imath},\vec{\jmath},\vec{k})$ are : Vx, Vy and Vz such that:

The vector$(\vec{V})$is written by:

$$\vec{V} = \vec{V}_x + \vec{V}_y + \vec{V}_z$$

$$\begin{cases} \vec{V}_x = V_x\vec{\imath} \\ \vec{V}_y = V_y\vec{\jmath} \\ \vec{V}_z = V_z\vec{k} \end{cases} \Rightarrow \vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$



The modulus of vector $\vec{V}$ *is calculated by :* $V = \sqrt{V_x^2 + V_y^2 + V_z^2}$

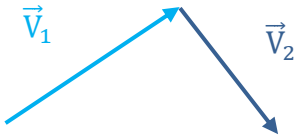
I.2.4. Elementary Operations on Vectors:

I.2.4.1. Vector addition:

If two vectors $\vec{V}_1$ and $\vec{V}_2$ are added, the initial point of the one vector is brought by a parallel shift to coincide with the endpoint of the other one. The sum $\vec{V}_1 + \vec{V}_2$, also called the resultant $\vec{V}_{12}$, then corresponds to the distance from the initial point of the first vector to the endpoint of the second one. This sum may also be found as the diagonal of the parallelogram formed by V_1 and V_2.

$$\vec{V}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}; \vec{V}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

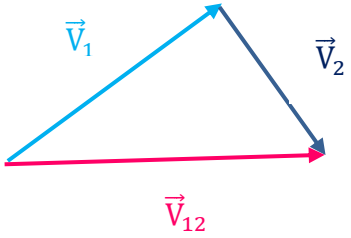
$\vec{V}_1 = \vec{V}_2$ if $x_1 = x_2$ and $y_1 = y_2$



The sum of the 2 vectors is a vector $\vec{S}$

$$\vec{S} = \vec{V}_1 + \vec{V}_2$$

$$\vec{S} = (x_1 + x_2)\vec{\imath} + (y_2 + y_2)\vec{\jmath}$$

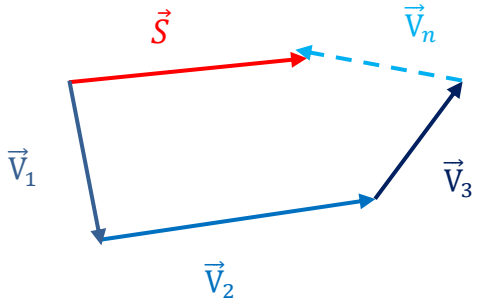


The modulus of $\vec{S}$ *is calculated by*:

$$S = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

The sum of n vectors $\vec{V}_1$,$\vec{V}_2$ and $\vec{V}_3$ …$\vec{V}_n$. Is a vector S such that:

$$\vec{S} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3 + \cdots \vec{V}_n$$



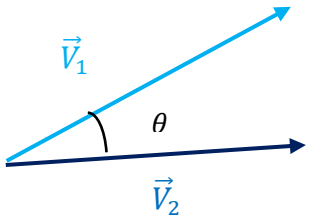
I.2.5. Products:

I.2.5.1. Scalar product:

The scalar product is a method for multiplying two vectors. Because the product of the multiplication is a scalar, the dot product will used to find an angle between two vectors and will have applications in finding distances between points and lines, points and planes, etc.

Scalar product between two vectors $\vec{V}_1$ and $\vec{V}_2$ with: $\vec{V}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}; \vec{V}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$

gives a scalar value: $\vec{V}_1 \cdot \vec{V}_2 = V_1 V_2 \cos(\vec{V}_1, \vec{V}_2)$



- ✓ If $\alpha = \frac{\pi}{2}, \cos\alpha = 0$ then $\vec{V}_1 \cdot \vec{V}_2 = 0$

Then the vector $\vec{V}_1$ is perpendicular to the vector $\vec{V}_2$

- ✓ If $\alpha = 0, \cos\alpha = 1$ then $\vec{V}_1 \cdot \vec{V}_2 = |\vec{V}_1| \cdot |\vec{V}_2| \cdot 1 = V_1 \cdot V_2$

Then the vector $\vec{V}_1$ is parallel to $\vec{V}_2$

$$\begin{cases} \vec{V}_1 \cdot \vec{V}_2 = \left(x_1\vec{\imath} + y_1\vec{\jmath}\right)\left(x_2\vec{\imath} + y_2\vec{k}\right) \\ \vec{V}_1 \cdot \vec{V}_2 = x_1x_2\vec{\imath}\vec{\imath} + x_1y_2\vec{\imath}\vec{\jmath} + x_2y_1\vec{\jmath}\vec{\imath} + y_1y_2\vec{\jmath}\vec{\jmath} \end{cases}$$

To simplify the calculation, we use the properties of the scalar product with respect to unit vectors:

$$\begin{cases} \vec{\imath}\cdot\vec{\imath} = 1, & \vec{\jmath}\cdot\vec{\jmath} = 1, & \vec{k}\cdot\vec{k} = 1 \\ \vec{\imath}\cdot\vec{\jmath} = 0, & \vec{\imath}\cdot\vec{k} = 0, & \vec{\jmath}\cdot\vec{k} = 0 \end{cases}$$

Using the commutative property of the scalar product, we get $\vec{V}_1 \cdot \vec{V}_2 = x_1x_2 + y_1y_2$

- The angle between the two will be given by:

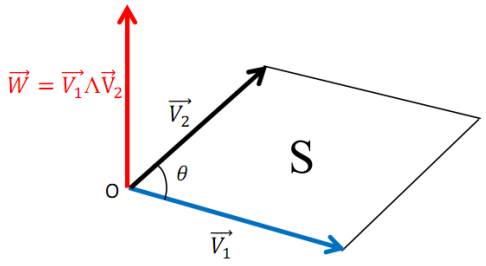
$$\cos\theta = \frac{x_1x_2 + y_1y_2 + z_1z_2}{|\vec{V}_1||\vec{V}_2|}$$

- The dot product allows us to define the magnitude of a vector:

$$|\vec{V}| = \sqrt{\vec{V}\cdot\vec{V}} = \sqrt{\vec{V}^2} = \sqrt{x^2 + y^2 + z^2}$$

I.2.5.2.Vector product:

$$\vec{w} = \|\vec{V}_1 \Lambda \vec{V}_2\| = \|\vec{V}_1\| \cdot \|\vec{V}_2\| \sin\theta$$



The modulus of vector $\vec{W}$ iscalculatedasfollows:

$$W = |\vec{V}_1 \times \vec{V}_2| = V_1V_2\sin(\vec{V}_1, \vec{V}_2)$$

The vector product can be calculated using the determinant method:

Let two vectors $\vec{V}_1$and $\vec{V}_2$ such that:

$$\vec{V}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\vec{W} = \vec{V}_1 \times \vec{V}_2 = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}$$

$$\vec{W} = (y_1z_2 - z_1y_2)\vec{\imath} - (x_1z_2 - z_1x_2)\vec{\jmath} + (x_1y_2 - y_1x_2)\vec{k}$$

From this relation, we can calculate the modulus of $\vec{W}$ by:

$$W = \sqrt{(y_1z_2 - z_1y_2)^2 + (x_1z_2 - z_1x_2)^2 + (x_1y_2 - y_1x_2)^2}$$

I.2.5.3. Mixed product :

The mixed product of three vectors:

$$\vec{V}_1 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}, \vec{V}_2 = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}, \vec{V}_3 = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$$

By the scalar product: $\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3)$

Which is calculates using the determinant method such that:

$$\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} x_1 - \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} y_1 + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} z_1$$

$$\vec{V}_1 \times (\vec{V}_2 \times \vec{V}_3) = (y_2 z_3 - z_2 y_3) x_1 - (x_2 z_3 - z_2 x_3) y_1 + (x_2 y_3 - y_2 x_3) z_1$$

Note:

The mixed products commutative: $\vec{V}_1 . (\vec{V}_2 \times \vec{V}_3) = \vec{V}_2 . (\vec{V}_1 \times \vec{V}_3) = (\vec{V}_1 \times \vec{V}_2)\vec{V}_3$

I.2.6. Derivative of a vector:

The derivative of vectors or vector-valued functions can defined similarly to the way we define the derivative of real-valued functions. Let us say we have the vector-values function,

$$\vec{\mathbf{V}}(t) = V_x(t)\vec{\mathbf{i}} + V_y(t)\vec{j} + V_z(t)\vec{\mathbf{k}}$$

Its first derivative in relation to time is:

$$\frac{d\vec{\mathbf{V}}(t)}{dt} = \frac{d\vec{\mathbf{V}}_x}{dt}\vec{\mathbf{i}} + \frac{d\vec{\mathbf{V}}_y}{dt}\vec{j} + \frac{d\vec{\mathbf{V}}_z}{dt}\vec{k}$$

$$\frac{d(\vec{\mathbf{V}}_1(t) + \vec{\mathbf{V}}_2(t))}{dt} = \frac{d\vec{\mathbf{V}}_1(t)}{dt} + \frac{d\vec{\mathbf{V}}_2(t)}{dt}$$

$$\frac{d(\lambda\vec{\mathbf{V}}(t))}{dt} = \frac{d\lambda}{dt}\vec{\mathbf{V}}(t) + \lambda\frac{d\vec{\mathbf{V}}(t)}{dt}$$

$$\frac{d(\vec{\mathbf{V}}_1(t) \cdot \vec{\mathbf{V}}_2(t))}{dt} = \frac{d\vec{\mathbf{V}}_1(t)}{dt}\vec{\mathbf{V}}_2(t) + \vec{\mathbf{V}}_1(t)\frac{d\vec{\mathbf{V}}_2(t)}{dt}$$

$$\frac{d(\vec{\mathbf{V}}_1(t) \wedge \vec{\mathbf{V}}_2(t))}{dt} = \frac{d\vec{\mathbf{V}}_1(t)}{dt} \wedge \vec{\mathbf{V}}_2(t) + \vec{\mathbf{V}}_1(t) \wedge \frac{d\vec{\mathbf{V}}_2(t)}{dt}$$

I.2.7. Differential operators:

I.2.7.1. Définitions:

A three − variable function is a function that depends on three variables x, y and z:

$F = f(x, y, z)$

The total differential of an algebraic function F with three variable x, y, z is written by:

$$dF = df(x, y, z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

With the partial differentials: $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$ $\frac{\partial f}{\partial z}$

I.2.7.2. Operators:

a) del: The vector differential operator del is de The vector differential operator del is defined as

$$\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

b) Gradient: The vector ∇f is called the gradient of f (written grad f).

The gradient is an operator that acts on algebraic functions and transforms them into vector functions by the del operator. We define the gradient vector of the algebraic function $f(x, y, z)$ as follows :

$$\overrightarrow{grad f} = \vec{\nabla} f = \left(\frac{\partial f}{\partial x}\right)\vec{\imath} + \left(\frac{\partial f}{\partial y}\right)\vec{\jmath} + \left(\frac{\partial f}{\partial z}\right)\vec{k}$$

c) Divergence: The divergent is an operator that acts on vector functions and transforms them into algebraic functions by the del operator.

Let it be $V = V_x\, i + V_y\, j + V_z\, k$ a vector function. We define $div\vec{V}$ as follows

$$div\vec{V} = \nabla \cdot \vec{V} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}\right)$$

$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

d) Curl : The Curl of the vector $\vec{v}$ is a vector that can be represented as $\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$

$$\vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$Curl\vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{\imath} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$

e) Laplacian: The following relation defines Laplacian of a scalar function:

$$\vec{\nabla}^2.(f) = \vec{\nabla}.\vec{\nabla}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The following relation gives Laplacian of a vector function:

$$\vec{\nabla}^2.(f) = \vec{\nabla}.\vec{\nabla}(f) = \frac{\partial^2 f}{\partial x^2}\vec{\imath} + \frac{\partial^2 f}{\partial y^2}\vec{\jmath} + \frac{\partial^2 f}{\partial z^2}\vec{k}$$

I.2.8. Moment of a vector to a point in space:

Definition: The moment of a vector with respect to a point in space is the vector defined by:

$$\vec{\tau}_o = \overrightarrow{OA} \wedge \vec{V}$$

I.2.9. Moment of a vector about an axis:

Definition: The moment of vector $\vec{v}$ about an axis ", with origin O and unit vector$\vec{u}$, is equal to the mixed product:

$$\boldsymbol{\tau_{\Delta} = \vec{\tau}_o \cdot \vec{u} = (\overrightarrow{OA} \wedge \vec{V}) \cdot \vec{u}}$$

$\vec{u}$: The unit vector of the axis (Δ).

Proposed exercises about chapter I

Part 1: Dimensional Analysis

Exercise 1:

- Find also dimensions of force F, acceleration a, energy W, pressure P, Work and Power.

- Find the dimensions of the following physical quantities: charge Q, surface density σ, electric field E, electric potential V, electrical resistance R, resistivity ρ, capacitance C

Exercise 2:

A physical relation must be dimensionally homogeneous, i.e., all the terms on both sides of the equation must have the same dimensions.

We can think that the period T of a simple pendulum could depend on the length l of the string, the mass m of the body and the acceleration due to gravity g.

Let us establish the relation, which describes this dependence.

Expression of T as a function of the previous quantities:

$T=Km^{\alpha}l^{\beta}g^{\gamma}$

K is a dimensionless constant and α, β and γ are exponents to be determined.

Exercise 3:

What are the dimensions of both k and v_0 constants in the following equation?

$v = kt + v_0$ where v represents velocity and t represents time.

Exercise 4:

The equation of state for an ideal gas is given by $PV = nRT$

- Write the dimensions of each parameter in this equation.

Exercise5:

The sound emitted by the wire of a guitar is characterized by its frequency f. This frequency is a function of the force F of the wire tension, the length L and the density ρ of the wire.

Find the expression of frequency f assuming the form:

$f=K\ F^{a}L^{b}\rho^{c}$

With K a dimensionless constant and the frequency dimension $[f]=T^{-1}$

Part 2: Vector Analysis

Exercise 6:

Let the vectors in space be $\vec{V_1} = 2\vec{\imath} - \vec{\jmath} + 3\vec{k}$ and $\vec{V_2} = -\vec{\imath} + \vec{\jmath} + 2\vec{k}$ represented in the frame $R(O,\vec{\imath}\vec{\jmath}\vec{k})$

-Calculate the angle between the two vectors $\vec{V_1}$ and $\vec{V_2}$.

Exercise 7:

1- Consider the following vectors: $\vec{V_1}$= sin t$\vec{\imath}$ +cos t$\vec{\jmath}$ +3t$\vec{k}$, $\vec{V_2} = e^{t}\vec{\imath}$ +2cos3t $\vec{\jmath}$ +3sin2t$\vec{k}$

- Find the derivatives: $\frac{d\vec{V_1}}{dt}$, $\frac{\overrightarrow{dV_2}}{dt}$

2- Let the vectors be the following:

$\vec{V}_1 = 5t^3\vec{\imath} + 3t\vec{\jmath} - 2t^4\vec{k}$, $\vec{V}_2 = \sin t\vec{\imath} + \cos t\vec{\jmath} + 3t\vec{k}$

- Find the following derivative expression in two ways: $\frac{d}{dt}(\vec{V}_1.\vec{V}_2), \frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2)$

Exercise 8:

A) Let be the points A (1, 1, 1), B (2, 2, 1) and C (2, 1, 0)

1. Calculate the scalar product $\overrightarrow{AB}, \overrightarrow{AC}$ and the vector product $\overrightarrow{AB}, \overrightarrow{AC}$

2. 2- Deduce the angle θ formed by the two vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

B) Consider the points A (1, 2, 3), B (3, 1, 3), C (8, 5, 5) and D (3, 5, 4) in space.

Calculate the mixed product by the three vectors ($\overrightarrow{AD}$,$\overrightarrow{BD}$, $\overrightarrow{DC}$).

Exercise 9:

In the frame $R\ (O,\vec{\imath}\vec{\jmath}\vec{k})$ we give the sliding vector $\vec{V} = \vec{\imath} + 2\vec{\jmath} + 3\ \vec{k}$ and which passes through the point A (3, 4, 2).

1. Calculate the moment of the vector $\vec{V}$ relative to the origin *O*, then relative to the axes OX and OY.

2. Calculate the moment of vector $\vec{V}$ relative to point B (3, 6, and 0)

3. Consider the (Δ) axis of unit vector $\vec{u}$ (-1/√2, 1/2, 1/2) and passing through *B*, calculate the moment of $\vec{V}$ relative to (Δ).

Exercises correction of about chapter I

Exercise1:

- **dimensions of:**

- Accélération a: [a] =[v]/[t]= $L.T^{-2}$

- Force F: [F] = [m]. [a]= [F] =$M.L.T^{-2}$ (kg.m.s^{-2} or Newton)

- Energy W: W=Force × distance = F. l, [W] = [F]. [l]= $M.L.T^{-2}$ .L= $M^1L^2T^{-2}$ (Joule)

- Pressure P: P = Force/area = F/S → [P] = [F]/[S] = $M.L.T^{-2}$ /L^2 = $M1L^{-1}T^{-2}$ (kg.m^{-1}.s^{-2} or Pascal).

- Work: W = F×d×cosα [W] = [F] × [d] × [cosα] = MLT^{-2}.L. 1=ML^2T^{-2} (kg.m^2.s^{-2} or Joule)

- Power: P = W/t [P] = [W]/ [t] = (ML^2T^{-2})/T=ML^2T^{-3} (kg.m^2.s^{-3} or Watt)

- **Dimension of physical quantities:**

- Electric charge: Q = current × time = i.t → [Q] = [i]. [t] = I.T

- Surface density: σ = charge/surface =Q/S → [σ] = [Q]/[S] = I.T/L^2 = I.T.L^{-2}

- Electric field: E= force/charge= F/Q → [E] = [F]/ [Q],

F= m.a → [F] = [m]. [a], [a]=[v]/[t]= $L.T^{-2}$ → [F] =$M.L.T^{-2}$ →[E] = [F]/[Q]= ($M.L.T^{-2}$)/ I.T

[E]= M^1 L^1 $T^{-3}I^{-1}$

- Electric potential: V= Electric field × distance = E.l or V= work/charge=W/Q

[V] = [E]. [l]= M^1. L^1 $T^{-3}I^{-1}$.L= M^1 L^2 T^{-3} I^{-1}

- Electric Resistance: R = Potential/current =V/I → [R] = [V]/ [i] = (M^1 L^2 T^{-3} I^{-1})/I

[R]= M^1 L^2 T^{-3} I^{-2}

- Resistivity: ρ = (Resistance surface)/length=R.S/l → [ρ] = [R]. [S]/ [l] = (M^1 L^2 T^{-3} I^{-2}).L^2/L

[ρ]= M^1 L^3 T^{-3} I^{-2}

- Capacitance: C= Charge/Potential = Q/V → [C] = [Q]/ [V] = I.T/ M^1 L^2 T^{-3} I^{-1}

[C]= M^{-1} L^{-2} T^4 I^2

Exercise2:

Now, dimensions of various quantities in the equation are: $[T]=T$, $[m^\alpha]=M^\alpha$, $[l^\beta]=L^\beta$, $[g^\gamma]=(L.T^{-2})^\gamma=L^\gamma.T^{-2\gamma}$

The dimensional equation is then: $T=M^\alpha.L^\beta.L^\gamma.T^{-2\gamma}$

Here the equation must be dimensionally homogeneous; the dimensions of all the terms on both sides of the equation must be the same. There for the following relations result: $\alpha=0$

$\beta+\gamma=0 \Leftrightarrow \beta=-\gamma$

$-2\gamma=1 \Leftrightarrow \gamma=-\frac{1}{2} \Rightarrow \beta=\frac{1}{2}$

The relation becomes: $T=K\sqrt{\frac{l}{g}}$

This analysis shows that the period of the pendulum does not depend on the mass m.

Exercise 3:

The equation is dimensionally correct if both sides have the same dimension.

$[v] = [k]\,[t] = [v_0]$

$LT^{-1} = [k]\,T = [v_0]$

$LT^{-1} = [v_0]$ *and* $LT^{-2} = [k]$

Exercise 4:

1) Pressure: $P = F/S$ (F: force and S: area)

$[P] = \frac{[F]}{[S]} \Rightarrow \quad F = m\,a \Rightarrow [F] = [m]\,[a]$

$S = L \times W \Rightarrow [S] = [L] \times [W]$ (m signifies mass, a represents acceleration, L denotes length, and w represents width.)

$[F] = [m]\,[a] = M\,L\,T^{-2}$

$[S] = [L] \times [W] = L^2$

$\Rightarrow [P] = \frac{[F]}{[S]} = \frac{MLT^{-2}}{L^2} = M\,L^{-1}\,T^{-2}$

2) Volume: $V = L \times W \times H \Rightarrow [V] = [L] \times [W] \times [H] = L^3$

3) Amount of substance: $[n] = N$

4) Temperature: $[T] = \theta$

5) The ideal gas constant, also known as the universal gas constant

$R = \frac{PV}{nT} \Rightarrow [R] = \frac{[PV]}{[nT]}$

$[n] = N, [T] = \theta$

$= \frac{M\,L^{-1}\,T^{-2}\,L^3}{N\,\theta}$

$\Rightarrow R = M\,L^2\,T^{-2}\,N^{-1}\theta^{-1}$

Exercise 5:

f = K $F^a L^b \rho^c$This function is therefore homogeneous $[f] = [K][F]^a[L]^b[\rho]^c$

$[F] = [m]\,[a] = M\,L\,T^{-2}$

$[L] = [K] = 1$

$[\rho] = \left[\frac{m}{v}\right] = ML^{-3}$

$[f] = \frac{1}{[T]} = T^{-1}$

So;

$[f] = (M\,L\,T^{-2})^a (L)^b (ML^{-3})^c = T^{-1}$

$\Rightarrow M^0 L^0 T^{-1} = M^{a+c} L^{a+b-3c} T^{-2a}$

By identification: $\begin{cases} a + b = 0 \\ a + b - 3c = 0 \\ -2a = -1 \end{cases}$

$$\begin{cases} a = \frac{1}{2} \\ b = -\frac{4}{2} = -2 \\ c = -a = -\frac{1}{2} \end{cases}$$

$$F = KF^{1/2}L^{-2}\rho^{-1/2} = K\sqrt{F}\frac{1}{L^2}\frac{1}{\sqrt{\rho}}$$

So; $f = K\frac{\sqrt{F}}{L^2\sqrt{\rho}}$

Exercise 6:

We have $\vec{V}_1.\vec{V}_2 = \|\vec{V}_1\|.\|\vec{V}_2\| cos\theta$

$$\Rightarrow cos\theta = \frac{\vec{V}_1.\vec{V}_2}{\|\vec{V}_1\|.\|\vec{V}_2\|}$$

$$\vec{V}_1.\vec{V}_2 = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}.\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = 3$$

$$\|\vec{V}_1\| = \sqrt{x_1^2 + y_1^2 + z_1^2} = \sqrt{4+1+9} = \sqrt{14}$$

$$\|\vec{V}_2\| = \sqrt{x_2^2 + y_2^2 + z_2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$cos\theta = \frac{\vec{V}_1.\vec{V}_2}{\|\vec{V}_1\|.\|\vec{V}_2\|} = \frac{3}{\sqrt{14}\sqrt{6}} = 0.32$$

$$\Rightarrow \boldsymbol{\theta = 71.33}$$

Exercise7:

- Find the derivatives: $\frac{d\vec{V}_1}{dt}, \frac{\overrightarrow{dV_2}}{dt}$

$\vec{V}_1$= sin t $\vec{\imath}$ +cos t $\vec{\jmath}$ +3t $\vec{k}$ $\Rightarrow \frac{d\vec{V}_1}{dt} =$ cos t $\vec{\imath}$ + sin t $\vec{\jmath}$ +3t $\vec{k}$

$\vec{V}_2 = e^t\vec{\imath}$ +2 cos3t $\vec{\jmath}$ +3 sin 2t $\vec{k}$ $\Rightarrow \frac{\overrightarrow{dV_2}}{dt} = e^t\vec{\imath} - 6$ sin 3t $\vec{\jmath}$ + 6 cos 2t $\vec{k}$

- Find the following derivative expression in two ways:

$$\frac{d}{dt}(\vec{V}_1.\vec{V}_2), \frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2)$$

1) $\vec{V}_1 = 5t^3\vec{\imath}$ +3t$\vec{\jmath}$ $-2t^4\vec{k} \Rightarrow \frac{d}{dt}(\vec{V}_1.\vec{V}_2) = \frac{d\vec{V}_1}{dt}.\vec{V}_2 + \vec{V}_1.\frac{d\vec{V}_2}{dt}$

2) $\vec{V}_2$= sin t$\vec{\imath}$ +cos t$\vec{\jmath}$ +3t$\vec{k} \Rightarrow \frac{d}{dt}(\vec{V}_1 \wedge \vec{V}_2) = \frac{d\vec{V}_1}{dt} \wedge \vec{V}_2 + \vec{V}_1 \wedge \frac{d\vec{V}_2}{dt}$

Exercise8:

A) Let be the points A (1, 1, 1), B (2, 2, 1) and C (2, 1, 0)

1. Calculate the scalar product $\overrightarrow{AB}.\overrightarrow{AC}$

$$\overrightarrow{AB}\begin{pmatrix}2-1\\2-1\\1-1\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix} \text{ And } \overrightarrow{AC}\begin{pmatrix}2-1\\1-1\\0-1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$$

$$\overrightarrow{AB}.\overrightarrow{AC}=\begin{pmatrix}1\\1\\0\end{pmatrix}.\begin{pmatrix}1\\0\\-1\end{pmatrix}=1$$

The vector product $\overrightarrow{AB}\wedge\overrightarrow{AC}$ with $\overrightarrow{AC}\begin{pmatrix}2-1\\1-1\\0-1\end{pmatrix}=\begin{pmatrix}1\\0\\-1\end{pmatrix}$

$$\overrightarrow{AB}\wedge\overrightarrow{AC}=\begin{vmatrix}\vec{\imath} & -\vec{\jmath} & \vec{k}\\1 & 1 & 0\\1 & 0 & -1\end{vmatrix}=-\vec{\imath}-(-1)\vec{\jmath}-\vec{k}=-\vec{\imath}+\vec{\jmath}-\vec{k}$$

$$|\mathrm{AB}|=|\mathrm{AC}|=\sqrt{2}$$

The angle between $\overrightarrow{AB}.\overrightarrow{AC}=\|\overrightarrow{AB}\|.\|\overrightarrow{AC}\|\cos\theta$

$$\cos\theta=\frac{\overrightarrow{AB}.\overrightarrow{AC}}{\|\overrightarrow{AB}\|.\|\overrightarrow{AC}\|}=\frac{1}{2}$$

$\Rightarrow \boldsymbol{\theta=\pi/3}$

B) Consider the points A (1, 2, 3), B (3, 1, 3), C (8, 5, 5) and D (3, 5, 4) in space.

Calculate the mixed product by the three vectors $(\overrightarrow{AD},\overrightarrow{BD},\overrightarrow{DC})$.

$$\overrightarrow{AD}\begin{pmatrix}3-1\\5-2\\4-3\end{pmatrix}=\begin{pmatrix}2\\3\\1\end{pmatrix},\ \overrightarrow{BD}\begin{pmatrix}3-3\\5-1\\4-3\end{pmatrix}=\begin{pmatrix}0\\4\\1\end{pmatrix} \text{ and } \overrightarrow{DC}\begin{pmatrix}8-3\\5-5\\5-4\end{pmatrix}=\begin{pmatrix}5\\0\\1\end{pmatrix}$$

$$\overrightarrow{AD}\wedge\overrightarrow{BD}=\begin{vmatrix}\vec{\imath} & -\vec{\jmath} & \vec{k}\\2 & 3 & 1\\0 & 4 & 1\end{vmatrix}=-\vec{\imath}+2\vec{\jmath}+8\vec{k}$$

$$(\overrightarrow{AD}\wedge\overrightarrow{BD}).\overrightarrow{DC}=\begin{pmatrix}-1\\2\\8\end{pmatrix}.\begin{pmatrix}5\\0\\1\end{pmatrix}=-3$$

Exercise 9:

Vector $\vec{V}=\vec{\imath}+2\vec{\jmath}+3\vec{k}$ which passes through the point A (3, 4, 2).

1) The moment of the vector $\vec{V}$ relative to the origin *O*

$$\vec{M}_{\vec{V}/O}=\overrightarrow{OA}\wedge\vec{V}$$

$$\overrightarrow{OA}\begin{pmatrix}x_A-x_O\\y_A-y_O\\z_A-z_O\end{pmatrix}\Rightarrow\overrightarrow{OA}\begin{pmatrix}3-0\\4-0\\2-0\end{pmatrix}=\begin{pmatrix}3\\4\\2\end{pmatrix}$$

$$\vec{M}_{\vec{V}/O} = \overrightarrow{OA} \wedge \vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 3 & 4 & 2 \\ 1 & 2 & 3 \end{vmatrix} = (12-4)\vec{\imath} - (9-2)\,\vec{\jmath} + (6-4)\vec{k} = 8\,\vec{\imath} - 7\vec{\jmath} + 2\vec{k}$$

$$\vec{M}_{\vec{V}/O} = \overrightarrow{OA} \wedge \vec{V} = 8\,\vec{\imath} - 7\vec{\jmath} + 2\vec{k}$$

Moment of $\vec{V}$ relative toOX:

$$\vec{M}_{\vec{V}/OX} = \vec{M}_{\vec{V}/O}.\vec{\imath} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 8$$

Moment of $\vec{V}$ relative toOY:

$$\vec{M}_{\vec{V}/OY} = \vec{M}_{\vec{V}/O}.\vec{\jmath} = \begin{pmatrix} 8 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 0$$

3) Moment of $\vec{V}$ relative to point B (3, 6, 0):

$$\vec{M}_{\vec{V}/B} = \overrightarrow{BA} \wedge \vec{V}$$

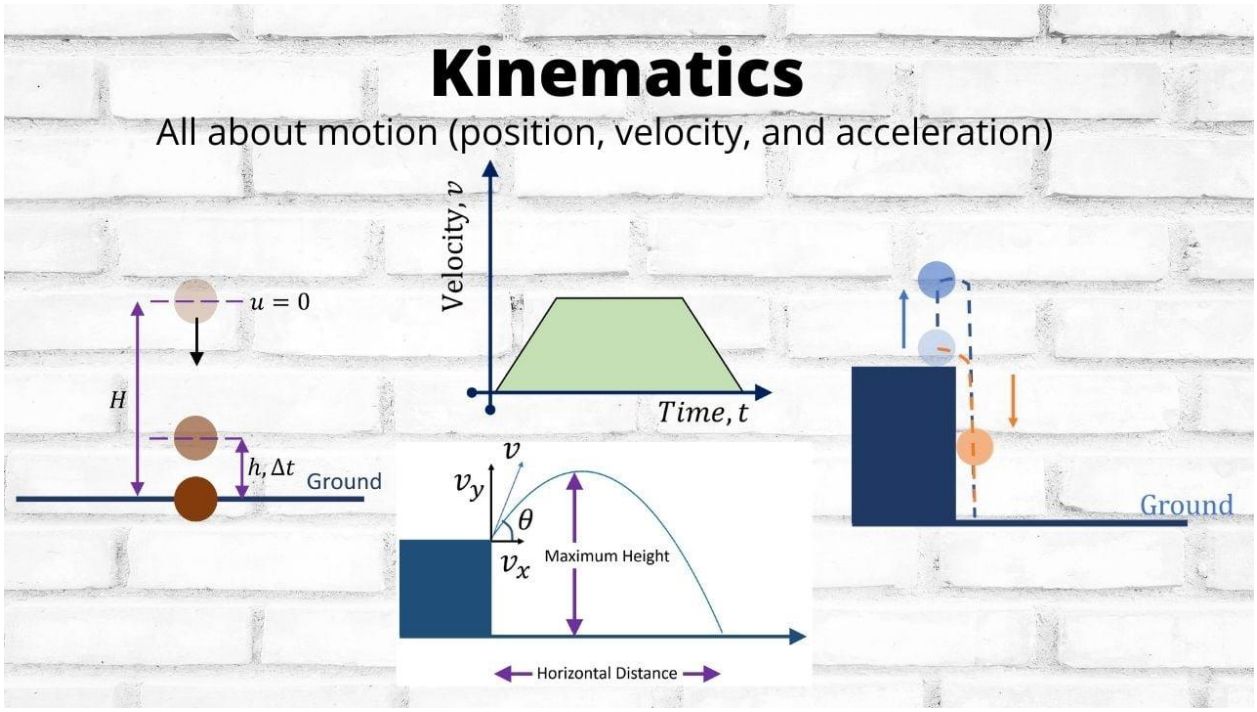
$$\overrightarrow{OA}\begin{pmatrix} x_A - x_B \\ y_A - y_B \\ z_A - z_B \end{pmatrix} \Rightarrow \overrightarrow{BA}\begin{pmatrix} 3-3 \\ 4-6 \\ 2-0 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 2 \end{pmatrix}$$

$$\vec{M}_{\vec{V}/B} = \overrightarrow{BA} \wedge \vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 0 & -2 & 2 \\ 1 & 2 & 3 \end{vmatrix} = = -10\,\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$$

4) Moment of $\vec{V}$ relative to Δ

$$\vec{M}_{\vec{V}/\Delta} = \vec{M}_{\vec{V}/B}.\vec{u} = \begin{pmatrix} -10 \\ 2 \\ 2 \end{pmatrix}.\begin{pmatrix} -1/\sqrt{2} \\ 1/2 \\ 1/2 \end{pmatrix} = \frac{10}{\sqrt{2}} + 2$$

CHAPTER II: KINEMATICS OF MATERIAL POINT



Chapter II: Kinematics of material point

II.1. Generality:

Kinematics is the branch of mechanics that describes the motion of an object. It is the study of the motion of objects without reference to the forces that caused the motion most bodies studied by physicists are in motion. Motion occurs at all scales of the universe, from particles like electrons, protons, and neutrons that constitute atoms, to galaxies. It is crucial to define motion accurately in order to understand many phenomena that we observe around us. An object can have motion in various forms: Translation, rotation and vibration.

II.2. Position of the moving object:

II.2.1. Position vector:

The position of a material point at a given time t is as defined by a vector$\overrightarrow{OM}$, where O is a fixed point and serves as the origin of the reference frame. The components of point M or the vector $\overrightarrow{OM}$ is as given in the chosen coordinate system's basis (Cartesian coordinates, polar coordinates, etc.).

$$\overrightarrow{\boldsymbol{OM}} = \vec{\mathbf{r}} = \boldsymbol{x}(\boldsymbol{t})\vec{\mathbf{i}} + \boldsymbol{y}(\boldsymbol{t})\vec{\mathbf{j}} + \boldsymbol{z}(\boldsymbol{t})\vec{\mathbf{k}}$$

II.2.2. Trajectory:

It is the sum of the consecutive points occupied by the mobile. To find the path equation, we look for a mathematical relationship between the representations *x* (*t*), *y* (*t*), and *z* (*t*) independent of time.

Example 1:

The position of a matrial point M identified by its coordinates x, y and z at time t in a coordinate system R $(O, \vec{\mathrm{i}}, \vec{\mathrm{j}}, \vec{\mathrm{k}})$with a position vector:

$$\overrightarrow{OM} = (t-1)\vec{\mathrm{i}} + \frac{t^2}{2}\vec{\mathrm{j}}$$

$$\overrightarrow{OM} = (t-1)\vec{\mathrm{i}} + \frac{t^2}{2}\vec{\mathrm{j}} \Rightarrow \begin{cases} \mathrm{x} = \mathrm{t} - 1 \\ \mathrm{y} = \dfrac{t^2}{2} \end{cases}$$

$So\ t = x + 1$

The trajectory equation of the material points $y = \frac{(x+1)^2}{2}$

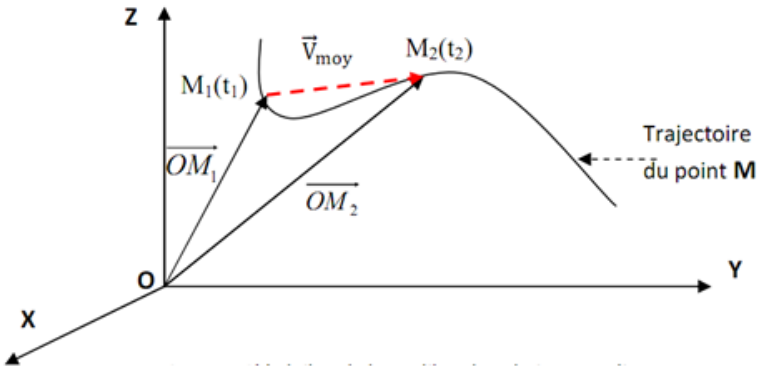
II.3. Velocity vector:

II.3.1. Average velocity:

The average velocity $\vec{v}_{\mathrm{ave}}$ of amoving M between the moments t_1 and t_2 is defined by the following relation:

$$\vec{v}_{ave} = \frac{\overrightarrow{M_1M_2}}{t_2 - t_1} = \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{\Delta t}$$

$$\Rightarrow \vec{v}_{ave} = \frac{\overrightarrow{OM_2}(t_2) - \overrightarrow{OM_1}(t_1)}{t_2 - t_1}$$



For the Cartesian coordinates, the analytical expression is given for the a verage velocity

$$\begin{cases} \overrightarrow{OM_1} = x_1\vec{\imath} + y_1\vec{\jmath} + z_1\vec{k} \\ \overrightarrow{OM_2} = x_2\vec{\imath} + y_2\vec{\jmath} + z_2\vec{k} \end{cases}$$

$$\vec{v}_{ave} = \left(\frac{x_2 - x_1}{t_2 - t_1}\right)\vec{\imath} + \left(\frac{y_2 - y_1}{t_2 - t_1}\right)\vec{\jmath} + \left(\frac{z_2 - z_1}{t_2 - t_1}\right)\vec{k}$$

II.3.2. Instantaneous velocity vector:

If thetimeinterval Δt is very small($\Delta t \to 0$), wethen refer to it as instantaneous velocity:

$$\vec{v} = \lim_{\Delta t \to 0} \vec{v}_{ave} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \frac{d\overrightarrow{OM}}{dt}$$

In a Cartesian coordinate system, we have

$$\overrightarrow{OM} = \vec{r} = x.\vec{\imath} + y.\vec{\jmath} + z.\vec{k} \Rightarrow \vec{v} = \dot{x}.\vec{\imath} + \dot{y}.\vec{\jmath} + \dot{z}.\vec{k}$$

In addition, we calculate the magnitude of the instantaneous velocity vector using the relation

$$\vec{v} = \frac{d\vec{r}}{dt} = \begin{cases} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{cases} \Rightarrow |\vec{v}| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

II.4. Acceleration vector:

When velocity varies over time v=f (t), point M is subjected to an acceleration.

II.4.1. Average acceleration vector:

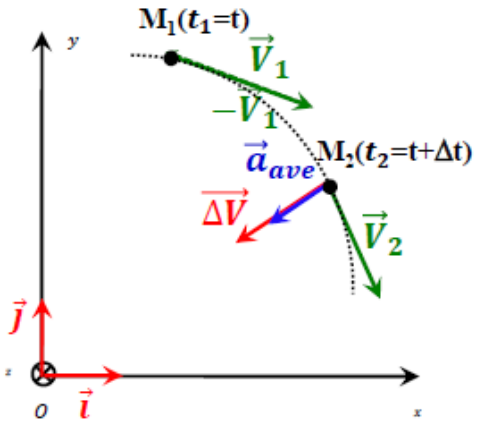
The average acceleration of the moving body between two positions is as given by:

$$\vec{a}_{ave} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_2 - \vec{v}_1}{t_2 - t_1}$$

In Cartesian coordinates:

$$\vec{V}_1\begin{pmatrix}\vec{V}_{x1}\\ \vec{V}_{y1}\\ \vec{V}_{z1}\end{pmatrix}, \vec{V}_2\begin{pmatrix}\vec{V}_{x2}\\ \vec{V}_{y2}\\ \vec{V}_{z2}\end{pmatrix}$$

$$\vec{a}_{ave} = \frac{\Delta V_x}{\Delta t}\vec{\imath} + \frac{\Delta V_y}{\Delta t}\vec{\jmath} + \frac{\Delta V_z}{\Delta t}\vec{k}$$



$$\vec{a}_{ave} = \frac{V_{x_2} - V_{x_1}}{t_2 - t_1}\vec{\imath} + \frac{V_{y_2} - V_{y_1}}{t_2 - t_1}\vec{\jmath} + \frac{V_{z_2} - V_{z_1}}{t_2 - t_1}\vec{k}$$

II.4.2. instantaneous acceleration vector:

The instantaneous acceleration of a moving body is the derivative of its velocity with respect to time at the instant under consideration:

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\overrightarrow{OM}}{\Delta t} \Rightarrow \vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

For the Cartesian coordinates,

$$\vec{a} = \ddot{x}.\vec{\imath} + \ddot{y}.\vec{\jmath} + \ddot{z}.\vec{k}$$

The magnitude of the instantaneous acceleration vector

$$|\vec{a}| = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2} = \sqrt{\left(\frac{dv_x}{dt}\right)^2 + \left(\frac{dv_y}{dt}\right)^2 + \left(\frac{dv_z}{dt}\right)^2}$$

II.5. Study of some movements:

II.5.1. Rectilinear motion:

Motion is rectilinear if the trajectory is a straight line. Described by a single coordinate and studied in a one-dimensional coordinate system (dimension). The expressions for the kinematics characteristics (the position, velocity and acceleration) of a moving particle given as follows:

$$\overrightarrow{OM} = x(t)\vec{\imath}$$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx(t)}{dt}\vec{\imath} = \dot{x}(t)\vec{\imath}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x(t)}{d^2t}\vec{\imath} = \ddot{x}(t)\vec{\imath}$$

II.5.2.Uniform rectilinear motion (URM):

Motion is uniform rectilinear (URM) if the moving object maintains the same speed throughout its entire path.

It is characterized by:

Straight – line trajectory (astraightline).

Constantvelocity(v = constant).

Zero acceleration (a = 0).

The initial conditions to $t = 0; x = x_0$.

- ***The velocity***

$$a = \frac{dv}{dt} = 0 \Rightarrow \int_{v_0}^{v} dv = \int_0^t 0.dt = cte$$

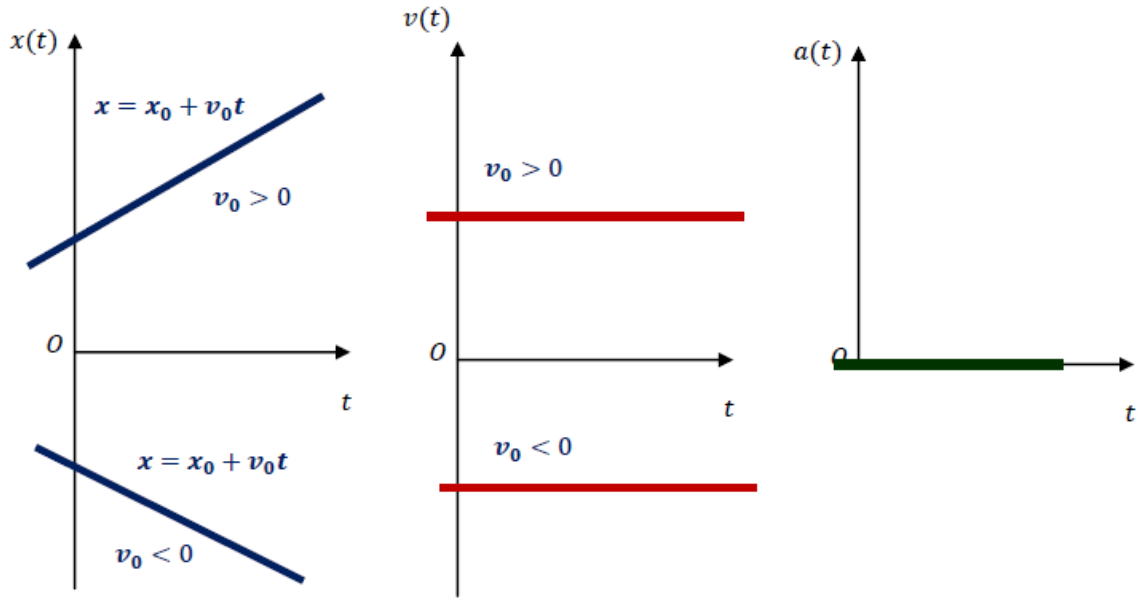
$So\ v = v_0 = \ cte$

- ***The position***

$$v = \frac{dx}{dt} \quad \Rightarrow dx = v \cdot dt$$
$$\int_{x_0}^{x} dx \quad = v\int_0^t dt$$
$$[x]_{x_0}^{x} \quad = v[t]_0^t$$
$$x - x_0 \quad = v(t-0)$$
$$x - x_0 \quad = vt \Rightarrow x = vt + x_0$$

So: $\boldsymbol{x = v_0 t + x_0}$

These diagrams are a way visually represents the motion of objects or particles over time. Motion diagrams can be particularly useful in physics to help understand and analyze the behavior of objects in motion.



II.5.3. Uniformly varied rectilinearmotion(UVRM):

The motion of a material point is rectilinear uniformly varied if its trajectory is a straight line, and its acceleration is constant. $\vec{a} = a_0\vec{\imath} = cste$

Considering the initial conditions:(t=0, V(0)= V_0).

- ***The velocity***

$$\frac{dv}{dt} = a_0 \Rightarrow \int_{v_0}^{v} dv = \int_{t_0}^{t} a_0 dt \Rightarrow v - v_0 = a_0(t - t_0)$$

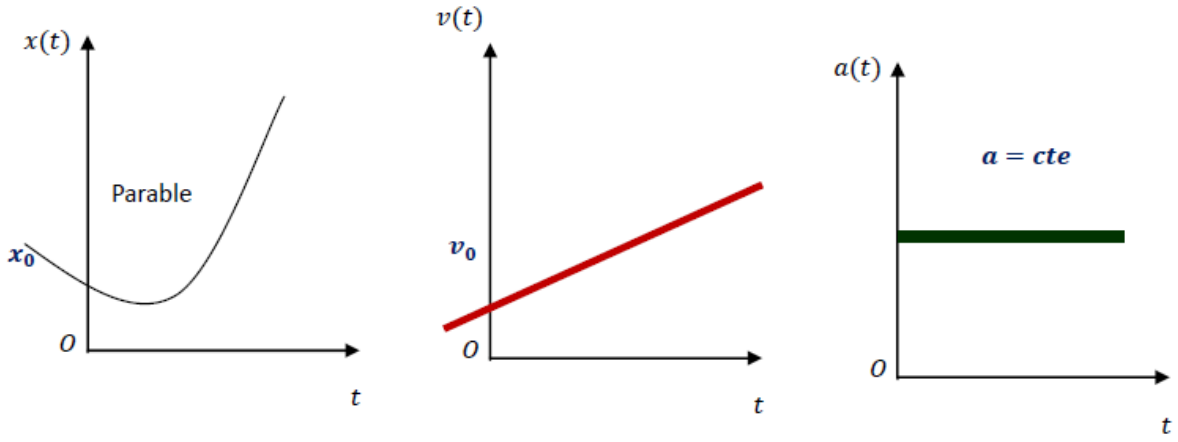
So $\boldsymbol{v = a_0 t + v_0}$

- ***The position***

$$v = \frac{dx}{dt} = a_0 t + v_0 \Rightarrow \int_{x_0}^{x} dx = \int_{0}^{t} (a_0 t + v_0)\, dt = \left[\frac{1}{2} a_0 t^2 + v_0 t\right]_0^t$$

So $\boldsymbol{x = \frac{1}{2} a_0 t^2 + v_0 t + x_0}$

The diagrams for uniformly accelerated rectilinear motion related to acceleration, velocity, and displacement are represents in the following figure:



The function $\boldsymbol{x(t)}$ is of the second degree, and the corresponding curve is a parabola.

Therefore, the equations of UVRM are:

$$\begin{cases} a = a_0 = cste \\ V(t) = (a_0 t + V_0) \\ x(t) = \frac{1}{2} a_0 t^2 + V_0 t + x_0 \end{cases}$$

II.5.3. Uniform, accelerated, and decelerated motion:

Three types of movement are distinguishes according to the evolution of the magnitude of the velocity vector.

a- Uniform motion: the magnitude of the velocity vector is constant.

$\vec{a} \cdot \vec{v} = 0 \Rightarrow a = 0$, **ou** $\alpha = \frac{\pi}{2}$

b- Accelerated motion: the magnitude of the velocity increases. $\vec{a} \cdot \vec{v} > 0$

c- Decelerated motion: the magnitude of the velocity decreases. $\vec{a} \cdot \vec{v} < 0$

II.5. Curvilinear motion:

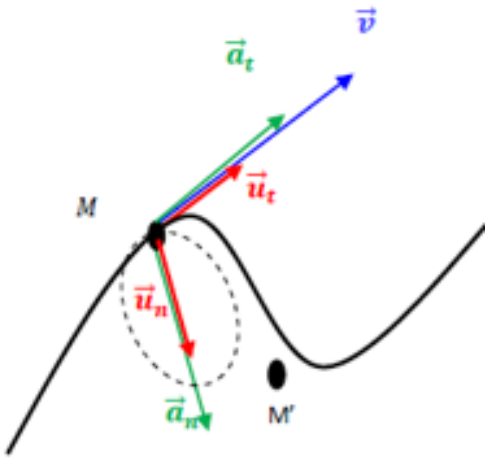
The curvilinear motion is characterized by a curvilinear trajectory which requires knowledge of the radius of curvature R and the center C.

II.5.1. Frenet basis (Intrinsic basis):

The base denotes the intrinsic coordinate system at for any point from the path $(\vec{u}_t, \vec{u}_N)$ where:

$(\vec{u}_t)$ Represent the normal unit vector and oriented towards the center of curvature.

$(\vec{u}_N)$ Represent the tangent unit vector and oriented in the direction of motion.



The velocity and acceleration vector is tangent to the trajectory and is as written:

$$\vec{V}(t) = \frac{dS(t)}{dt}\vec{u}_t$$

$$S(t) = R\theta(t) \Rightarrow \vec{V}(t) = R\frac{d\theta(t)}{dt}\vec{u}_t$$

$$\Rightarrow \vec{V}(t) = R\omega(t)\vec{u}_\theta = R\omega(t)\vec{u}_t$$

Where dS is the curvilinear abscissa.

For the **intrinsic coordinates**, we perform the same steps, only we replace the unit vectors as follows:

$\vec{u}_r = -\vec{u}_N$ and $\vec{u}_\theta = \vec{u}_t$

The acceleration vector is as written by:

$$\vec{a}(t) = a_N\vec{u}_N + a_t\vec{u}_t = \frac{dV(t)}{dt}\vec{u}_t + \frac{1}{\rho}V^2\vec{u}_N$$

$$\begin{cases} \rho = R \\ V = R\omega \\ \omega = \frac{d\theta}{dt} \end{cases} \Rightarrow \vec{a}(t) = R\frac{d\omega(t)}{dt}\vec{u}_t + R\omega^2(t)\vec{u}_N$$

$$By: \begin{cases} a_t = \frac{d|\vec{v}|}{dt} \\ a_N = \frac{v^2}{R} \end{cases} \Rightarrow \begin{cases} a_N = R\dot{\theta}^2 \\ a_t = R\ddot{\theta} \end{cases}$$

$$So, \quad |\vec{a}| = \sqrt{a_N^2 + a_t^2}$$

Where, $\vec{a}_t$ is the Tangential Acceleration. Associated with the changes of the magnitude of velocity.

$\vec{a}_N$ is the Normal Acceleration. Related to the change in direction of velocity.

Hence:

$\omega = \frac{v}{R}$ with R is the radius of the curvature of the trajectory.

II.6. Circular motion:

The circular motion is a movement whose trajectory is a circle of constant radius R. The equation of the trajectory is as follows:

R is constant, the speed is following the trajectory, so it is written = vu so the vector u would be following the tangent.

- ***Position vector***:

Since the trajectory is a circle, the object can be located using **polar coordinates**:

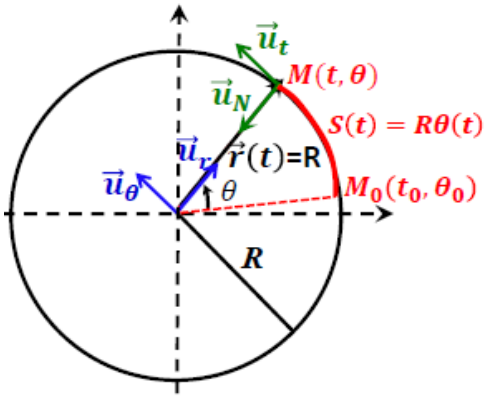
$$\overrightarrow{OM} = \begin{cases} r(t) = R = Constante \\ \theta = \theta(t) \end{cases}$$

$$\overrightarrow{OM} = \text{r}.\overrightarrow{u_r} = \text{R}.\overrightarrow{u_r}$$

- ***Velocity vector:***

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(R.\overrightarrow{u_r}) = \frac{dR}{dt}\overrightarrow{u_r} + R\frac{d\overrightarrow{u_r}}{dt} = R\dot{\theta}\overrightarrow{u_\theta}$$

$$\vec{v}(t) = R\dot{\theta}\overrightarrow{u_\theta}$$



$\boldsymbol{\frac{d\theta}{dt} = \dot{\theta} = \omega}$: *is the angular velocity*

$\boldsymbol{v = R\dot{\theta} = R\omega}$

- ***Acceleration vector:***

For the instantaneous **acceleration** vector in **polar coordinates** is as given follows:

$$\vec{a} = \frac{d\vec{v}(t)}{dt} = \frac{d}{dt}\left(R\dot{\theta}\,\vec{u}_\theta\right) = R\frac{d^2\theta}{dt^2}\vec{u}_\theta + R\dot{\theta}\frac{d\vec{u}_\theta}{dt}$$

We have:

$$\boldsymbol{\frac{d\vec{u}_\theta}{dt} = -\vec{u}_r\frac{d\theta}{dt}}$$

$$\boldsymbol{\vec{a} = \frac{d\vec{v}(t)}{dt} = -R\dot{\theta}^2\vec{u}_r + R\ddot{\theta}\vec{u}_\theta}$$

$\boldsymbol{\ddot{\theta}}$: is the angular acceleration.

II.6.1. Uniformly Circular motion:

In this case, the angular velocity ω is constant and therefore the linear velocity v is constant, then $a_T = 0$

V (t) = Cst = $R\omega$(t) ⇒ $\omega(t) = Cst$

$$\vec{a} = \begin{cases} a_t = R\dfrac{d\omega(t)}{dt} = 0 \\ a_N = R\omega^2 \end{cases}$$

Or

$$\vec{a} = \begin{cases} a_r = -R\omega^2(t) \\ a_\theta = R\dfrac{d\omega(t)}{dt} = 0 \end{cases}$$

The acceleration in this case is:

$$\boldsymbol{\vec{a} = \vec{a}_N = \frac{v^2}{R}\vec{n}}$$

II.6.2. Uniformly variable circular motion:

In this case, the angular velocity ω is not constant and therefore the velocity *v* is not constant then

$\vec{a} = a_T\vec{u} + a_N\vec{n}$

The acceleration in this case is:

$$\vec{a} = \frac{dv}{dt}\vec{u} + \frac{v^2}{R}\vec{n} = R\frac{d\omega}{dt}\vec{u} + R\omega^2\vec{n}$$

II.7. Expression of velocity and acceleration in different coordinate systems:

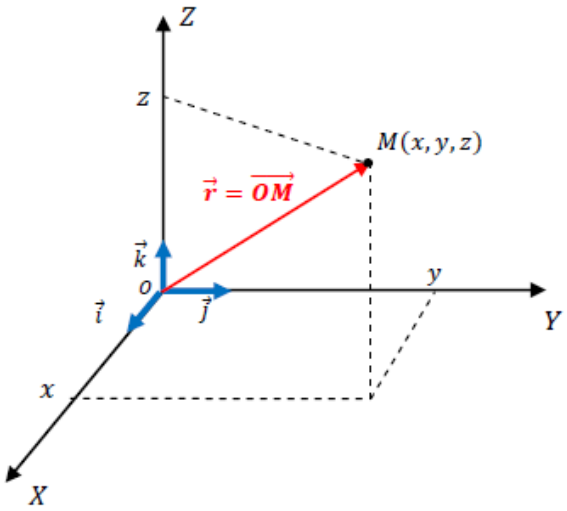
II.7.1.Cartesian coordinates:

R (x,y,z) is a director the normal coordinate system with base $(\vec{i}, \vec{j}, \vec{k})$ the point M is located by

$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = x\vec{I} + y\vec{J} + z\vec{K}$

The position of point M is given by its three Cartesian coordinates (x,y,z) such that

x = abscissa of M; y = ordinate of M; z = altitude of M



The vector of position, velocity and acceleration are written according to the cartesian coordinates as follows:

- ***Position vector:***

$\overrightarrow{OM} = x\vec{I} + y\vec{J} + z\vec{k}$

- ***Velocity vector:***

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

- ***Acceleration vector:***

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k} = a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

II.7.2. Polar coordinates:

We can locate the position of the point M by a coordinate r (distance between the origin of the mark and the point M) and θ oriented angle that the vector $\vec{r}$ makes with the abscissa axis (OX).

The data (r, θ) called polar coordinates, the basis of the polar coordinate system is formed by two unit vectors $(\vec{u}_r, \vec{u}_\theta)$

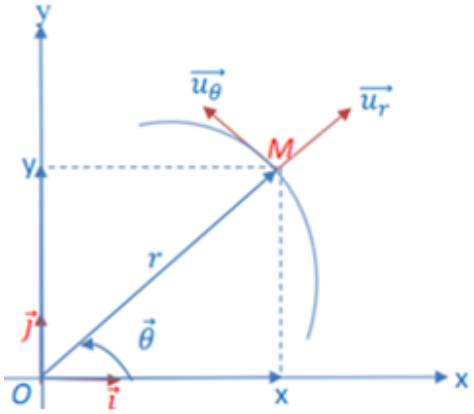
$$\overrightarrow{OM} = \vec{r} = r\vec{u}_r$$

$\vec{u}_r$: Unit vector following the direction of the position vector $\overrightarrow{OM}$
$\vec{u}_\theta$: Unit vector perpendicular to $\vec{u}_r$ by rotating $\pi/2$ of the vector $\overrightarrow{OM}$

Where:

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ r = \sqrt{x^2 + y^2} \end{cases}$$

$$\begin{cases} \vec{u}_r = \cos\theta\vec{i} + \sin\theta\vec{j} \\ \vec{u}_\theta = -\sin\theta\vec{i} + \cos\theta\vec{j} \end{cases}$$



- ***Velocity vector:***

To find expression for the velocity vector $\vec{v}$(t), we differentiate the expression for the posision vector $\overrightarrow{OM}$ with respect to time

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(r\vec{u}_r) = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt}$$

$$\frac{d\vec{u}_r}{dt} = -\vec{i}\cdot\sin\theta\cdot\frac{d\theta}{dt} + \vec{j}\cos\theta\cdot\frac{d\theta}{dt} = \vec{u}_\theta\cdot\frac{d\theta}{dt} \Rightarrow \frac{d\vec{u}_r}{dt} = \vec{u}_\theta\cdot\frac{d\theta}{dt}$$

$$\frac{d\vec{u}_\theta}{dt} = -\vec{j}\cdot\cos\theta\cdot\frac{d\theta}{dt} - \vec{j}\cdot\sin\theta\cdot\frac{d\theta}{dt} = -\vec{u}_r\cdot\frac{d\theta}{dt} \Rightarrow \frac{d\vec{u}_\theta}{dt} = -\vec{u}_r\frac{d\theta}{dt}$$

The velocity vector according the polar coordinates is as written by:

$$\vec{v} = \dot{\vec{r}} = r\frac{d\vec{u}_r}{dt} + \vec{u}_r\frac{dr}{dt} \Rightarrow \vec{v} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta \Leftrightarrow \vec{v} = \dot{r}\cdot\vec{u}_r + r\cdot\dot{\theta}\cdot\vec{u}_\theta$$

- ***Acceleration vevtor***

To find expression for the acceleration vector $\vec{a}$(t), we differentiate the expression for the velocity vector $\vec{v}$(t) with respect to time.

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

We have:

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta\right)$$

and $\vec{a}(t) = \frac{d^2r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{dt}$

Now, we calculate the expression $\frac{d\vec{u}_\theta}{dt}$

$$\frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}(-\sin(\theta)\vec{i} + \cos(\theta)\vec{j})$$

We have $\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt}(-\cos(\theta)\vec{i} - \sin(\theta)\vec{j}) = -\frac{d\theta}{dt}\vec{u}_r$

The acceleration vector according the polar coordinates is written by:

$$\vec{a} = \vec{v} = \dot{r}\cdot\frac{d\vec{u}_r}{dt} + \ddot{r}\cdot\vec{u}_r + r\cdot\dot{\theta}\cdot\frac{d\vec{u}_\theta}{dt} + r\cdot\ddot{\theta}\cdot\vec{u}_\theta + \dot{r}\cdot\dot{\theta}\cdot\vec{u}_\theta$$

$$\vec{a} = \dot{r}\cdot(\vec{u}_\theta\cdot\frac{d\theta}{dt}) + \ddot{r}\cdot\vec{u}_r + r\cdot\dot{\theta}\cdot(-\vec{u}_r\frac{d\theta}{dt}) + r\cdot\ddot{\theta}\cdot\vec{u}_\theta + \dot{r}\cdot\dot{\theta}\cdot\vec{u}_\theta$$

$$\vec{a} = \dot{r}\vec{u}_\theta\dot{\theta} + \ddot{r}\cdot\vec{u}_r + \dot{r}\cdot\dot{\theta}(-\vec{u}_r\dot{\theta}) + r\cdot\ddot{\theta}\cdot\vec{u}_\theta + \dot{r}\cdot\dot{\theta}\cdot\vec{u}_\theta$$

$$\vec{a} = (\dot{r} - r\cdot\dot{\theta}^2)\vec{u}_r + (2\dot{r}\cdot\dot{\theta} + r\cdot\ddot{\theta})\vec{u}_\theta$$

II.7.3. Cylindrical Coordinates:

Cylindrical coordinates correspond to polar coordinates in the (O,x, y) plane If we add the 'z' component to polar coordinates in space, we obtain what is known as cylindrical coordinates. Consider R(Oxyz) and a point M belonging to a cylinder.

Point M is identified by three coordinates ρ, θ (polar coordinates), and z.

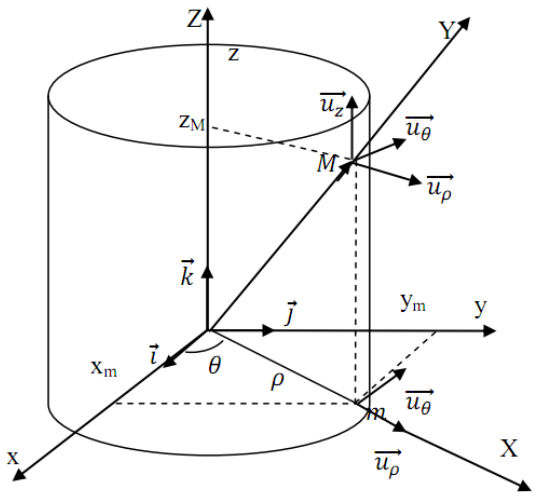
ρ: polar radius

θ: polar angle

z: altitude or height

Where m is the projection of point M onto the plane (Oxy), and R is the radius of the cylinder,

and m is the height of the cylinder.



Position vector The position vector in cylindrical coordinates (ρ, θ, z) in the orthonormal frame $R'(O, \vec{u}_\rho, \vec{u}_\theta, \vec{u}_z)$

is written :$\vec{\mathbf{r}} = \overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM}$

$$\Rightarrow \begin{cases} \overrightarrow{OM} = \rho\vec{u}_\rho + z\vec{u}_z \\ \overrightarrow{Om} = \rho\vec{u}_\rho \\ \overrightarrow{mM} = z\vec{u}_z \end{cases}$$

$$\begin{cases} \vec{u}_\rho = \cos\theta \cdot \vec{\imath} + \sin\theta \cdot \vec{\jmath} \\ \vec{u}_\theta = -\sin\theta \cdot \vec{\imath} + \cos\theta \cdot \vec{\jmath} \\ \vec{u}_z = \vec{k} \end{cases}$$

$$\boldsymbol{\overrightarrow{OM} = \vec{\imath} \cdot \rho \cdot \cos\theta + \vec{\jmath} \cdot \rho \cdot \sin\theta + z \cdot \vec{k}}$$

- ***Velocity vector:***

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$

$$\vec{r} = \rho \cdot \vec{u}_\rho + z \cdot \vec{u}_z \Rightarrow \vec{v} = \dot{\vec{r}} = \frac{d\vec{r}}{dt} = \vec{u}_\rho \frac{d\rho}{dt} + \rho \frac{d\vec{u}_\rho}{dt} + \vec{u}_z \frac{dz}{dt}$$

So, the vector of velocity is:

$$\vec{v} = \dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{u}_z$$

The magnitude of the velocity in cylindrical coordinates is given by the expression:

$$v = \sqrt{\dot{\rho}^2 + (\rho \cdot \dot{\theta})^2 + \dot{z}^2}$$

- ***Acceleration vector :***

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{\rho}\vec{u}_\rho + \rho\dot{\theta}\vec{u}_\theta + \dot{z}\vec{u}_z)$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{\rho}\cdot\vec{u}_\rho + \dot{\rho}\cdot\frac{d\vec{u}_\rho}{dt} + \dot{\rho}\cdot\dot{\theta}\cdot\vec{u}_\theta + \rho\cdot\ddot{\theta}\cdot\vec{u}_\theta + \rho\cdot\dot{\theta}\cdot\frac{d\vec{u}_\theta}{dt} + \ddot{z}\cdot\vec{u}_z$$

Applying Newton's notation, and recalling the following relationships:

$$\frac{d\vec{u}_\rho}{dt} = \vec{u}_\theta\frac{d\theta}{dt}, \qquad \frac{d\vec{u}_\theta}{dt} = -\vec{u}_\rho\frac{d\theta}{dt}$$

Then the expression for acceleration in cylindrical coordinates is given by:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{u}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\vec{u}_\theta + \ddot{z}\vec{u}_z$$

So the components of the acceleration vector in the cylindricalnbase are:

$$\begin{cases} a_\rho = \ddot{\rho} - \rho\dot{\theta}^2 \\ a_\theta = 2\dot{\rho}\dot{\theta} + r\ddot{\theta} \\ a_z = \ddot{z} \end{cases}$$

II.7.4. Spherical coordinates:

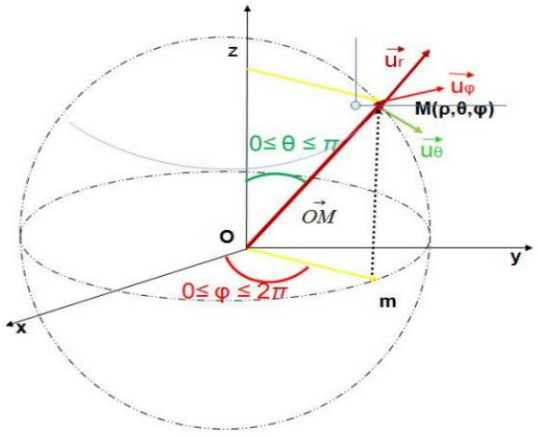
The unit vectors determine the spherical basis $(\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi)$ such that:

φ: is the angle between the position vector $\overrightarrow{OM}$ and the Oz axis.

θ: is the angle between the vector $\overrightarrow{OM}$ and the Ox axis.

m: is the projection of M in the plane (0,x,y).

$$\begin{cases} r = |\overrightarrow{OM}|, 0 < r < R \\ \theta = \left((ox), \overrightarrow{Om}\right) 0 < \varphi < 2\pi \\ \varphi = \left((oz), \overrightarrow{OM}\right) 0 < \theta < \pi \end{cases}$$



- ***Position vector:***

The position vector in spherical coordinates $(r, \theta, \varphi)$is written as:

$$\vec{r} = \overrightarrow{OM} = r\vec{u}_r$$

- ***Spherical basisasa function of the Cartesian basis***

$$\begin{cases} x = |\overrightarrow{Om}| \cos\theta \\ y = |\overrightarrow{Om}| \sin\theta \\ z = |\overrightarrow{mM}| \end{cases}$$

By taking the right triangle (OmM)

We have : $Om = r\sin\varphi$ and $mM = r\cos\varphi$,

replacing them in passing relation ships we will have

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

En utilisant les relations entre les vecteurs: $\vec{u}_r, \vec{u}_\theta, \vec{u}_\varphi$

$$\vec{u}_r = \sin\theta\cdot\cos\varphi\vec{\imath} + \sin\theta\cdot\sin\varphi\vec{\jmath} + \cos\vec{k}$$

$$\vec{u}_\varphi = -\sin\varphi\vec{\imath} + \cos\varphi\vec{\jmath}$$

$$\vec{u}_\theta = \cos\theta\cdot\cos\varphi\vec{\imath} + \cos\theta\cdot\sin\varphi\vec{\jmath} - \sin\vec{k}$$

- **Velocity vector:**

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}(r.\overrightarrow{u_r}) = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt}$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\overrightarrow{u_r}}{d\theta}\frac{d\theta}{dt} + \frac{d\overrightarrow{u_r}}{d\varphi}\frac{d\varphi}{dt}$$

$$\overrightarrow{\dot{u}_r} = \dot{\theta}\,[\underbrace{\cos\theta\cdot\cos\varphi\vec{\imath} + \cos\theta\cdot\sin\varphi\vec{\jmath} - \sin\vec{k}}_{\vec{u}_\theta}] + \dot{\varphi}\sin\theta[\underbrace{-\sin\varphi\vec{\imath} + \cos\varphi\vec{\jmath}}_{\vec{u}_\varphi}]$$

Avec:

$$\overrightarrow{\dot{u}_r} = \dot{\theta}\vec{u}_\theta + \dot{\varphi}\sin\theta\vec{u}_\varphi$$

Substituting in to the velocity equation, we obtain.

$$\vec{v} = \dot{r}\vec{u}_r + r\dot{\theta}\vec{u}_\theta + (r\sin)\dot{\varphi}\vec{u}_\varphi$$

Then the three components of the velocity vector appear as follows:

$$\vec{v} = \vec{v}_r + \vec{v}_\theta + \vec{v}_\varphi \Rightarrow \vec{v} = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta + r\sin\theta\frac{d\varphi}{dt}\vec{u}_\varphi$$

- ***Acceleration vector :***

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\dot{r}\vec{u}_r + (r\sin\varphi)\dot{\theta}\vec{u}_\theta + r\dot{\varphi}\vec{u}_\varphi\right]$$

$$\vec{a} = \begin{pmatrix} a_r \\ a_\theta \\ a_\varphi \end{pmatrix}, \left\|\vec{a}\right\| = \sqrt{a_r^2 + a_\theta^2 + a_\varphi^2}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta\right)\vec{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta\right)\vec{u}_\theta + \left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta\right)\vec{u}_\varphi$$

The acceleration components in the spherical basis are:

$$\vec{a} = \begin{cases} a_r = \ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta \\ a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta \\ a_\varphi = r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta \end{cases}$$

II.8. Relative Motion:

II.8.1. Change of Reference Frames:

The study of the movement of a particle in all of the above was in a fixed (absolute) frame of reference R.

We can also choose another frame of reference R' in motion relative to the first, and determine the position, velocity and acceleration of the mobile.

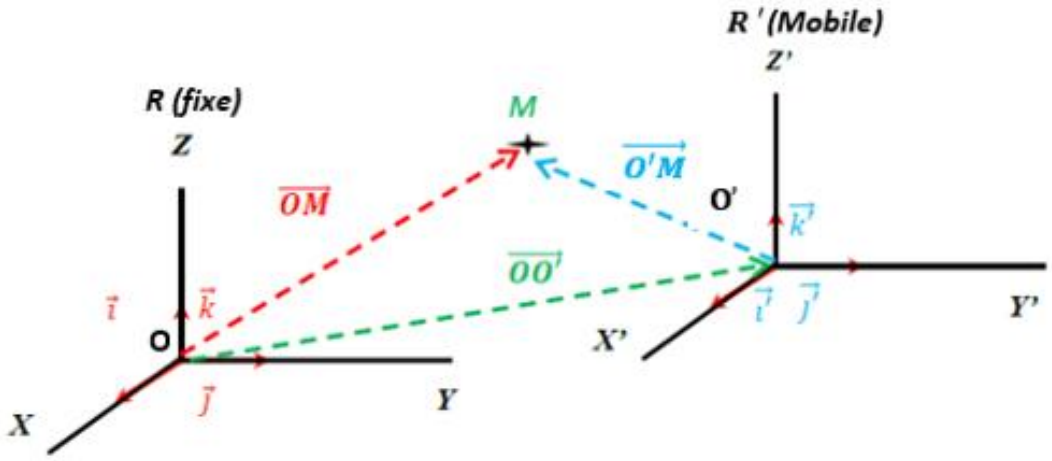
- $R\ (O, i, j, k)$ fixed reference, which is called absolute reference.

- $R'\ (O', i', j', k')$ in any movement relative to R which is called relative reference.

- The movement of point M relative to « R » is called absolute movement.

- The movement of « R' » relative to « R » is called entrainment movement.

To determine the vectors: position, speed and acceleration, we have different methods: direct method and composition method.



Representation of a relative motion

Let the first frame of reference R (O, xyz) defined by the vectors $(\vec{\imath},\vec{\jmath},\vec{k})$. We consider it fixed and call it the absolute frame of reference. In R, we define:

- ***position vector:***

$$\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

Absolute velocity: which is the velocity with respect to the reference R.

$$\vec{v}_a = \frac{d\overrightarrow{OM}}{dt} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k}$$

Absolute acceleration $\vec{a}_a$: which is the acceleration with respect to the reference R.

$$\vec{a}_a = \left(\frac{d\vec{v}_a}{dt}\right)$$

$$\Rightarrow \vec{a}_a = \frac{d^2\overrightarrow{OM}}{dt^2} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$$

Let the second frame of reference R' (O', x' y', z') be defined by the vectors$(\vec{\imath'},\vec{\jmath'},\vec{k'})$. It is in motion relative to R, so we call it the relative frame of reference. In R', we define:

- ***Position vector:***

$$\overrightarrow{O'M} = x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

- ***Relationship between positions:***

$$\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$$

$$x\vec{\imath} + y\vec{\jmath} + z\vec{k} = \overrightarrow{OO'} + x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}$$

- ***Relationship between speeds:***

$$\frac{d\overrightarrow{OM}}{dt} = \frac{d\overrightarrow{OO'}}{dt} + \frac{d\overrightarrow{O'M}}{dt}$$

$$\vec{v}_a = \frac{d\overrightarrow{OO'}}{dt} + \frac{d}{dt}\left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$$

$$\Rightarrow \underbrace{\frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}}_{\vec{V}_a(t)\text{: Absolute Velocity}} = \underbrace{\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}}_{\vec{V}_r(t)\text{ : Relative Velocity}} + \underbrace{\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\vec{V}_e(t)\text{ : Training Velocity}}$$

II.8.2. Theorem of Velocity Composition:

The absolute velocity vector is equal to the sum of the entrainment and relative velocity vectors:

$$\vec{v}_a(M) = \vec{v}_r + \vec{v}_e$$

$$\frac{d\vec{\imath'}}{dt} = \frac{d\vec{\jmath'}}{dt} = \frac{d\vec{k'}}{dt} = 0 \Rightarrow \vec{V}_e(t) = \frac{d\overrightarrow{OO'}}{dt}$$

- ***Relationship between accelerations :***

$$\frac{d}{dt}\left(\frac{dx}{dt}\vec{\imath}+\frac{dy}{dt}\vec{\jmath}+\frac{dz}{dt}\vec{k}\right)=\frac{d}{dt}\left(\frac{dx'}{dt}\vec{\imath'}+\frac{dy'}{dt}\vec{\jmath'}+\frac{dz'}{dt}\vec{k'}\right)+\frac{d}{dt}\left(\frac{d\overrightarrow{OO'}}{dt}+x'\frac{d\vec{\imath'}}{dt}+y'\frac{d\vec{\jmath'}}{dt}+z'\frac{d\vec{k'}}{dt}\right)$$

$$\frac{d^2x}{dt^2}\vec{\imath}+\frac{d^2y}{dt^2}\vec{\jmath}+\frac{d^2z}{dt^2}\vec{k}=\frac{d^2x'}{dt^2}\vec{\imath'}+\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt}+\frac{d^2y'}{dt^2}\vec{\jmath'}+\frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt}+\frac{d^2z'}{dt^2}\vec{k'}+\frac{dz'}{dt}\frac{d\vec{k'}}{dt}$$
$$+\frac{d^2\overrightarrow{OO'}}{dt^2}+\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt}+x'\frac{d^2\vec{\imath'}}{dt^2}+\frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt}+y'\frac{d^2\vec{\jmath'}}{dt^2}+\frac{dz'}{dt}\frac{d\vec{k'}}{dt}+z'\frac{d^2\vec{k'}}{dt^2}$$

$$\vec{a}_a=\frac{d^2x}{dt^2}\vec{\imath}+\frac{d^2y}{dt^2}\vec{\jmath}+\frac{d^2z}{dt^2}\vec{k}$$ ***Absolute acceleration***

$$\vec{a}_r=\frac{d^2x'}{dt^2}\vec{\imath'}+\frac{d^2y'}{dt^2}\vec{\jmath'}+\frac{d^2z'}{dt^2}\vec{k'}$$ ***Relative acceleration***

$$\vec{a}_e=\frac{d^2\overrightarrow{OO'}}{dt^2}+x'\frac{d^2\vec{\imath'}}{dt^2}+y'\frac{d^2\vec{\jmath'}}{dt^2}+z'\frac{d^2\vec{k'}}{dt^2}$$ ***Training acceleration***

$$\vec{a}_C=2\left(\frac{dx'}{dt}\frac{d\vec{\imath'}}{dt}+\frac{dy'}{dt}\frac{d\vec{\jmath'}}{dt}+\frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$ ***Coriolis acceleration***

II.8.3. Theorem of Composition of Accelerations:

The absolute acceleration vector is equal to the sum of the entrainment, relative, and Coriolis acceleration vectors:

$$\vec{a}_a=\vec{a}_r+\vec{a}_e+\vec{a}_c$$

$\vec{a}_e$: represents the ***entrainment acceleration***, is the acceleration of a point A, fixed in the moving reference frame $\Re'$, as observed from the absolute reference frame $\Re$. It takes into account the second derivative of the position vector of the point coinciding with A with respect to time.

$\vec{a}_r$: ***Relative acceleration*** represents the acceleration of a point M as observed from the moving reference frame $\Re'$.

$\vec{a}_c$: ***Coriolis acceleration*** represents an additional acceleration component caused by the change in the orientation of the moving reference frame $\Re'$. It arises due to the relative motion between the reference frames and often encountered in systems with rotation.

Remarks:

This relationship remains valid for any vector, in particular for unit vectors. $\vec{\imath},\vec{\jmath}$ and $\vec{k}$ We then write:

$$\frac{d\vec{\imath}}{dt}=\vec{\omega}\wedge\vec{\imath},\qquad\frac{d\vec{\jmath}}{dt}=\vec{\omega}\wedge\vec{\jmath}\ and\ \frac{d\vec{k}}{dt}=\vec{\omega}\wedge\vec{k}$$

$\boldsymbol{\omega}$ is called the rotational velocity vector; it corresponds to the rotation of the axes of R' with respect to an axis whose direction is defined by that of ω. Note that ω does not correspond to the rotation of O' in R.

$$\vec{V}_e = \underbrace{\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}}_{\text{Translation}} = \frac{d\overrightarrow{OO'}}{dt} + \underbrace{x'\vec{\omega}\wedge\vec{i'} + y'\vec{\omega}\wedge\vec{j'} + z'\vec{\omega}\wedge\vec{k'}}_{\text{Rotational}}$$

$$\vec{V}_e = \frac{dO\overrightarrow{O'}}{dt} + \vec{\omega}\wedge(x'\vec{\imath} + y'\vec{\jmath} + z'\vec{k})$$

$$\vec{V}_e = \frac{d\overrightarrow{OO'}}{dt} + \vec{\omega}\wedge\overrightarrow{O'M}$$

$$\vec{a}_C = 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right) = 2\left(\frac{dx'}{dt}\vec{\omega}\wedge\vec{i'} + \frac{dy'}{dt}\vec{\omega}\wedge\vec{j'} + \frac{dz'}{dt}\vec{\omega}\wedge\vec{k'}\right)$$

$$= 2\vec{\omega}\wedge\left(\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}\right)$$

$$\vec{a}_C = 2\vec{\omega}\wedge\vec{V}_r$$

For $\vec{a}_e$: $\frac{d^2\vec{i'}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{i'}}{dt}\right) = \frac{d}{dt}\left(\vec{\omega}\wedge\vec{i'}\right) = \frac{d\vec{\omega}}{dt}\wedge\vec{i'} + \vec{\omega}\wedge\frac{d\vec{i'}}{dt} = \frac{d\vec{\omega}}{dt}\wedge\vec{i'} + \vec{\omega}\wedge(\vec{\omega}\wedge\vec{i'})$

Special cases:

- If M fixed in R', $\vec{v}_r = \vec{0} \Rightarrow \vec{v}_a(M) = \vec{v}_e$
- If R' is fixed relative to R, $\vec{v}_e = \vec{0} \Rightarrow \vec{v}_a(M) = \vec{v}_r$
- If R' in translation movement relative to R, $\vec{\omega} = \vec{0} \Rightarrow \vec{v}_e = \frac{\overrightarrow{OO'}}{dt}$
- If R' in rotational movement relative to R $\Rightarrow \vec{v}_e = \vec{\omega}_{(R'/R)}\wedge\overrightarrow{O'M}$

Proposed exercises about chapter II

Exercise 1:

The motion of a point M is as given by the following time equations:

$x = t$

$y = 2t^2 + 1$

z = −3t

- Determine the vectors of velocity and acceleration of particle M.

Exercise 2:

A particle is in motion and is accelerating, the functional form of the velocity is:

$V(t) = 20t - 5t^2$ (m/s)

1- Find the functional form of the acceleration.

2- Find the instantaneous velocity at t = 1, 2, 3, and 5 s.

3- Find the instantaneous acceleration at t = 1, 2, 3, and 5 s.

Exercise 3:

The position of a particle in (oxy) plan as a function of time in SI units is:

$\vec{r}$ $\{x=t$ and $y=\frac{1}{2}t^2\}$

1- Find the trajectory equation then plot it.

2- Write expressions for its velocity and its acceleration and deduce the magnitude of the both as functions of time.

3- Find the intrinsic components (tangential and normal) of acceleration.

Exercise 4:

The position of a particle in (oxy) plan as a function of time in SI units is:

$$\begin{cases} x = t \\ y = \frac{1}{2}t^2 \end{cases}$$

1- Find the trajectory equation then plot it.

2- Write expressions for its velocity and its acceleration and deduce the magnitude of the both as functions of time.

3- Find the intrinsic components (tangential and normal) of acceleration.

Exercise 5:

The motion of a body is as defined by the following velocity components:

$$\begin{cases} v_x = R\omega\cos(\omega t) \\ v_y = R\omega\sin(\omega t) \end{cases}$$

Knowing that ω is constant and at t= 0, the moving body is at point M (0, R).

Determine:

1. The components of the acceleration vector and its modulus.
2. The tangential and normal components of acceleration, and deduce the radius of curvature.
3. The components of the position vector and deduce the equation of the trajectory.
4. What is the nature of the motion?

Correction of exercises about chapter II

Exercise 1:

1) Velocity:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k} = v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$

$$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k} = t\vec{i} + 2t^2\vec{j} + 3t\vec{k} \Rightarrow \vec{v} = \begin{cases} v_x = \frac{dx}{xt} = 1 \\ v_y = \frac{dy}{dt} = 4t \\ v_z = \frac{dz}{dt} = -3 \end{cases}$$

$$\vec{v} = \vec{i} + 4t\vec{j} - 3\vec{k}$$

2) Acceleration:

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{v}_x\vec{i} + \dot{v}_y\vec{j} + \dot{v}_z\vec{k}$$

$$\Rightarrow \vec{a} = \begin{cases} v_x = \frac{dx}{xt} = 0 \\ v_y = \frac{dy}{dt} = 4 \\ v_z = \frac{dz}{dt} = 0 \end{cases}$$

$$\vec{a} = 4\vec{j}$$

Exercise 2:

1- The acceleration is the derivative of velocity with respect to time:

$a\ (t) = dV dt = 20 - 10t$

2- The instantaneous velocity at different times: Given velocity function:

$V\ (t) = 20t - 5t^2$ (m/s)

At t=1s: V (1) =20(1) −5 $(1)^2$ =15 m/s

At t=2s: V (2) =20(2) −5 $(2)^2$ =20 m/s

At t=3s: V (3) =20(3) −5 $(3)^2$ =15 m/s

At t=5s: V (5) =20(5) −5 $(5)^2$ =−25 m/s

3- The instantaneous acceleration at different times:

$a\ (t) = 20 - 10t$ (m/s^2)

At t=1s: a (1) =20−10 (1) =10= 10 m/s^2

At t=2s: a (2) =20−10 (2) =0 a (2) = 0 m/s²

At t=3s: a (3) =20−10 (3) = −10 m/s²

At t=5s: a (5) =20−10 (5) =−30= −30 m/s²

Exercise 3:

1- The trajectory equation: $y = f(x) \Leftrightarrow y = \frac{1}{2}x^2$, the trajectory is a parabola .

2- Velocity and acceleration:

$$\vec{V}\begin{cases} V_x = \dfrac{dx}{dt} = 1 \\ V_y = \dfrac{dy}{dt} = t \end{cases} \qquad \Rightarrow |\vec{V}| = \sqrt{1+t^2}$$

$$\vec{a}\begin{cases} a_x = \dfrac{dV_x}{dt} = 0 \\ a_y = \dfrac{dV_y}{dt} = 1 \end{cases} \qquad \Rightarrow |\vec{a}| = 1 \ (m.s^{-2})$$

3- The intrinsic components (tangential and normal) of acceleration:

- Tangential accelerationa_T :

$$a_T = \frac{d|\vec{V}|}{dt} = \frac{t}{\sqrt{1+t^2}}$$

- Normal accelerationa_N :

$$\vec{a} = \overrightarrow{a_T} + \overrightarrow{a_N} = a_T\overrightarrow{u_T} + a_N\overrightarrow{u_N}$$

$$a^2 = {a_T}^2 + {a_N}^2 \Rightarrow a_N = \sqrt{a^2 - {a_T}^2} = \sqrt{1 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2} = \frac{1}{\sqrt{1+t^2}}$$

Exercise 4:

1) - The trajectory equation: $y=f\,(x) \Leftrightarrow y = \frac{1}{2}x^2$, the trajectory is a parabola.

2) - Velocity and acceleration:

$$\vec{V}\begin{cases} V_x = \dfrac{dx}{dt} = 1 \\ V_y = \dfrac{dy}{dt} = t \end{cases} \qquad \Rightarrow |\vec{V}| = \sqrt{1+t^2}$$

$$\vec{a}\begin{cases} a_x = \dfrac{dV_x}{dt} = 0 \\ a_y = \dfrac{dV_y}{dt} = 1 \end{cases} \qquad \Rightarrow |\vec{a}| = 1 \ (m.s^{-2})$$

3) - The intrinsic components (tangential and normal) of acceleration:

- Tangential acceleration a_T:

$$a_T = \frac{d|\vec{V}|}{dt} = \frac{t}{\sqrt{1+t^2}}$$

- Normal acceleration a_N:

$$\vec{a} = \overrightarrow{a_T} + \overrightarrow{a_N} = a_T\overrightarrow{u_T} + a_N\overrightarrow{u_N}$$

$$a^2 = {a_T}^2 + {a_N}^2 \Rightarrow a_N = \sqrt{a^2 - {a_T}^2} = \sqrt{1 - \left(\frac{t}{\sqrt{1+t^2}}\right)^2} = \frac{1}{\sqrt{1+t^2}}$$

Exercise 5:

1) The components of the acceleration vector and its modulus

$$\begin{cases} a_x = \frac{dv_x}{dt} = -R\omega^2 sin\,(\omega t) \\ a_y = \frac{dv_y}{dt} = R\omega^2 \cos(\omega t) \end{cases}$$

$$\lceil\vec{a}\rceil = \sqrt{(-R\omega^2 sin\,(\omega t))^2 + (R\omega^2 cos\,(\omega t))^2} = R\omega^2$$

2) The tangential and normal components of acceleration and deduce the radius of curvature.

- Tangential acceleration:

$$\lceil\vec{v}\rceil = \sqrt{(R\omega cos(\omega t))^2 + (R\omega sin\,(\omega t))^2} = R\omega$$

$$a_T = \frac{dv}{dt} = \frac{dR\omega}{dt} \Rightarrow a_T = 0$$

- Normal acceleration:

$$a_N = \frac{v^2}{R} = a = R\omega^2 \quad and \quad a_T = 0$$

- Radius of curvature is R:

$$R = \frac{v^2}{a_N} = \frac{R^2\omega^2}{R\omega^2} = \mathrm{R}$$

3) The components of the position vector:

$$\begin{cases} v_x = R\omega\cos(\omega t) \\ v_y = R\omega\sin(\omega t) \end{cases} \Rightarrow \begin{cases} \frac{dx}{dt} = R\omega\cos(\omega t) \\ \frac{dy}{dt} = R\omega\sin(\omega t) \end{cases}$$

$$\Rightarrow \begin{cases} dx = R\omega\cos(\omega t)dt \\ dy = R\omega\,\sin\,(\omega t)\,dt \end{cases}$$

$$\Rightarrow \begin{cases} \int dx = R\int \omega\cos(\omega t)dt \\ \int dy = R\int \omega \sin(\omega t)\, dt \end{cases}$$

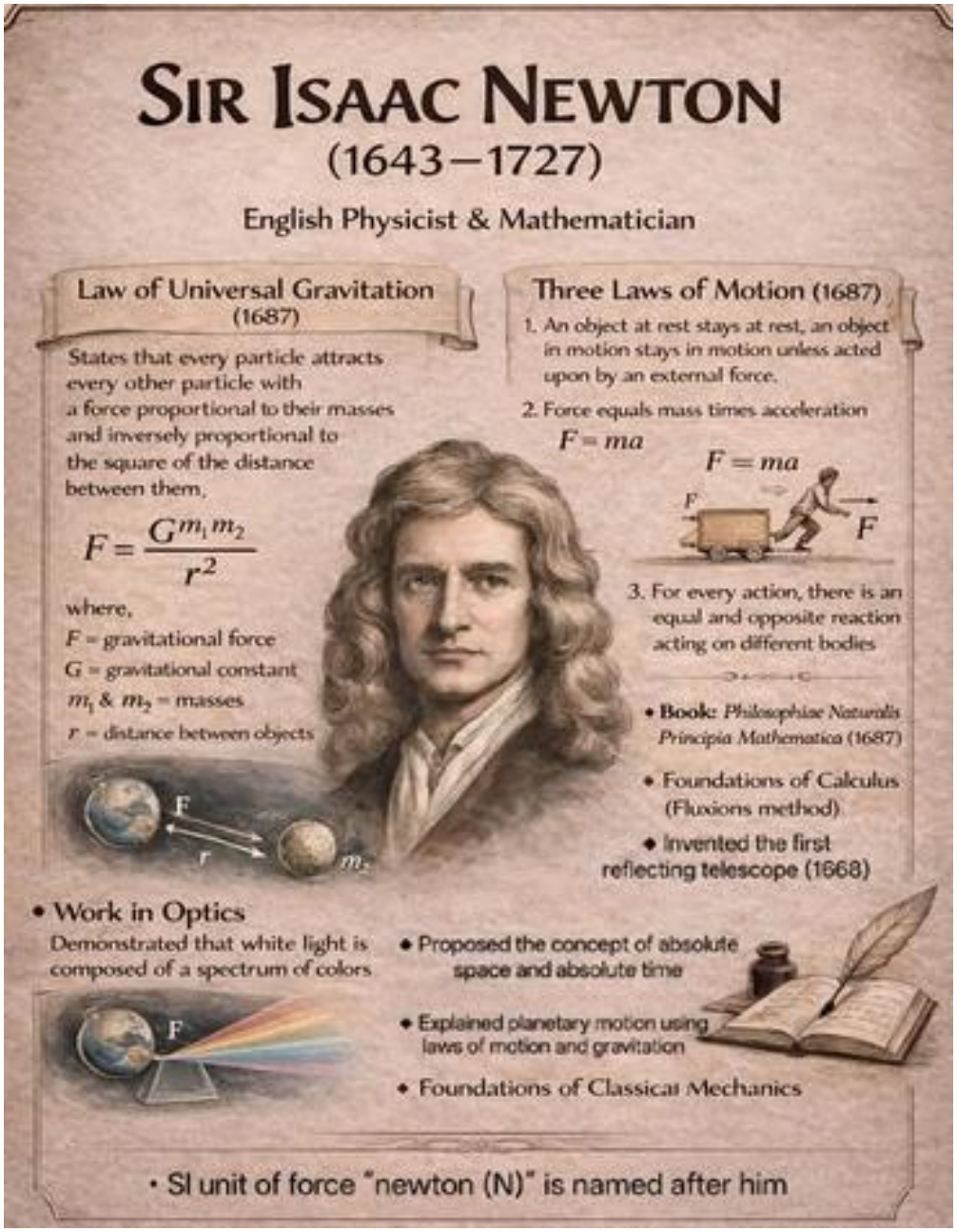
$$\Rightarrow \begin{cases} x = R\sin(\omega t) \\ y = -R\cos(\omega t) \end{cases}$$

- The trajectory equation:

$$x^2 + y^2 = R^2 sin^2\omega t + R^2 cos^2\omega t \Rightarrow x^2 + y^2 = R^2$$

4) The nature of movement: The motion is uniformly circular.

CHAPTER III: DYNAMICS OF A PARTICLE



Chapter III: Dynamics of a particle

III.1. Generalities:

In physics, dynamics is the science that studies the relationship between a body in motion and the causes of that motion. It also predicts the motion of a body located in a given environment. Dynamics, more precisely, is the analysis of the relationship between applied force and changes in body motion.

III.1.1. Concept of Force:

Motion is the result of the interaction between the particle and its environment. This interaction, called force, is characterized by the properties of the particle (mass, charge, dipole moment, etc.) and by the nature of the environment in which it is placed. We have seen that in physics, force is intuitively perceives as a quantity that reflects interactions between objects. It is a cause capable of producing or modifying the motion of a body, or causing its deformation.

There are two main categories of forces:

A- Contact forces: friction forces, tension forces, etc.

B- Forces at a distance: gravitational forces, electric forces, magnetic forces.

III.2. Principle of Inertia (Newton's First Law):

III.2.1. Free Particle:

A free particle is one that is not subject to any interaction. Such a particle does not truly exist because it must be alone in the universe! (Isolated particle). Nevertheless, we can assume that a particle sufficiently far from others is isolated. If the resultant of the interactions applied to a particle is zero, we say that it is pseudo-isolated; it behaves like a free particle.

III.2.2. Statement of the Principle of Inertia:

Galileo was the first to suggest this principle. It constitutes Newton's first law and is as stated as follows:

"Any free particle remains at rest or in uniform rectilinear motion in the absence of any forces acting upon it."

This first law can also state as follows: $\sum \vec{F} = \vec{0}$

If no force acts on an object, or if the net force is zero,

III.2.3. Inertial or Galilean reference frame:

The reference frame of the earth is not Galilean because of its orbital motion and its movement around the sun and its own rotation around its axis. However, in the vast majority of experiments, it is as considered a Galilean reference frame because we do studies with low times.

III.3. Newton's laws of motion:

3.1. The momentum:

The vector given by defines the momentum vector of a point mass m moving at velocity v

$$\vec{P} = m\vec{v}$$

The unit of momentum in the International System of Units is the kg.m.s^{-1}.

The principle of inertia can then state is as follows:

"A free particle moves with a constant momentum in a Galilean frame of reference."

III.4. Newton's three laws:

III.4.1. Galilean principle of inertia (Newton's first laws):

Newton's first law, also known as the law of inertia, states that any object continues to move at a constant speed in a straight line, or remains at rest, unless an external force is applied to it. In other words if the material body is not subjected to any force, it is either in uniform rectilinear motion, or at rest, if it was initially at rest.

For a particle, the principle of inertia thus states: A free and isolated particle moves in rectilinear motion with constant velocity".

III.4.2. Newton's second law (Fundamental Principle of Dynamics):

In a Galilean frame of reference, the sum of the external forces applied to a system is equal to the derivative of the momentum vector of the center of mass of that system.

$$\vec{F} = \frac{d\vec{P}}{dt}$$

This equation is as called the "equation of motion»

$$\vec{F} = \frac{dm\vec{v}}{dt} \Rightarrow \vec{F} = m\frac{d\vec{v}}{dt} + \vec{v}\underbrace{\frac{dm}{dt}}_{=0} = m\frac{d\vec{v}}{dt}$$

So, $\vec{F} = m\vec{a}$

This is because the mass m of the moving particle is constant (as is often the case in Newtonian mechanics).

In general, Newton's second law for a moving particle can written as:

$$\sum \vec{F}_{ext} = m\vec{a}$$

In the S.I. system, the unit of force is 1Newton=1N=1kg.m.s^{-2}.

III.4.3. Newton's third law or principle of action and reaction:

A force acting on an object is always due to another object in the surrounding environment.

Newton's third law shows that: if body 1 exerts a force, an $F_{2/1}$ on body 2then body will exert an equal and opposite force $F_{1/2}$ on body 1. That is:

$$\frac{d\vec{p}_1}{dt} = \vec{F}_{21}\,, \qquad \frac{d\vec{p}_2}{dt} = \vec{F}_{12} \Rightarrow \vec{F}_{12} = -\vec{F}_{21}$$



III.5. Classification of forces:

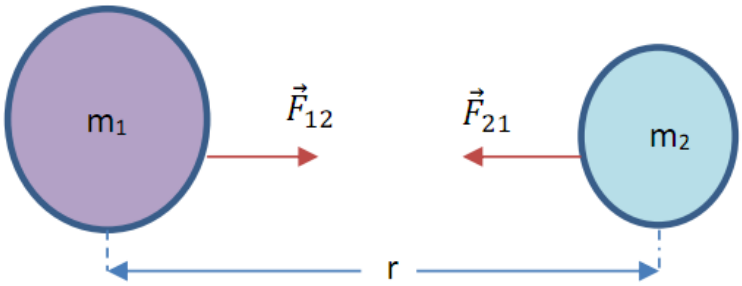
III.5.1. Forces at a distance:

The body, which exerts the force, is not in contact with the one on which it acts. There are three kinds of forces at a distance:

III.5.1.1. Law of Gravity:

Newton's law of gravity states that every particle in the universe attracts every other particle with a force that is directly proportional to the product of the masses of the particles and inversely proportional to the square of the distance between them. The magnitude of this gravitational force is as given by:

$$\|\overrightarrow{F_{12}}\| = \|\overrightarrow{F_{21}}\| = G\frac{m_1 m_2}{r^2}$$



where G is a constant, the value of which is experimentally determined to be:

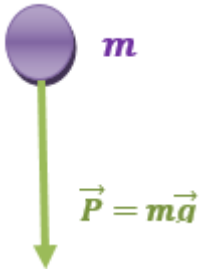
G: the gravitational constant, G = $6.67.10^{-11}$ [$m^3/kg.s^2$]

Near the earth, the force of gravitation is what keeps objects on the ground.

III.5.1.2. Weight of a mass:

Consider a point mass m, in gravitational interaction with Earth. The latter acts on the mass with a force that we called the weight of the mass. Newton's second law allows us to define this weight:

$$\vec{P} = m\vec{g}$$



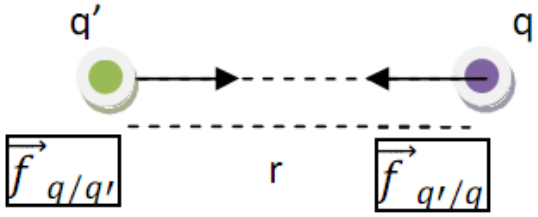
$\vec{g}$ Is the acceleration of gravity (terrestrial acceleration), g=9.80 m/s^2.

III.5.1.3. Electric force:

Consider two electric charges q and q' separated by a distance r. The electric force exerted by q on q' is given by:

$$\vec{F}_{q'q} = \frac{1}{4\pi\varepsilon_0}\frac{q'q}{r^2} = k\frac{q'q}{r^2}$$

$$k = \frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 (USI)$$



K: Is the universal electrostatic constant, q and q' the two interacting charges.

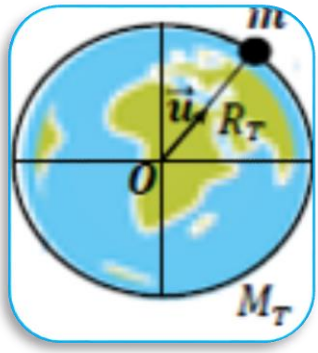
III.5.1.4. Gravitation on the surface of a star:

We can then specify what is called the weight of an object force that acts on a body. Between two bodies of masses m and M, distant from r, it force of attraction of magnitude

$$F_g = m \cdot g(r) = m \cdot \left(\frac{GM_T}{r^2}\right)$$

$$P = m \cdot g_0 = m \cdot \left(\frac{GM_T}{R_T^2}\right)$$

$$g_0 = \frac{GM_T}{R_T^2}$$



M_T : is the mass of the Earth, equal to5.98 10^{34} Kg.

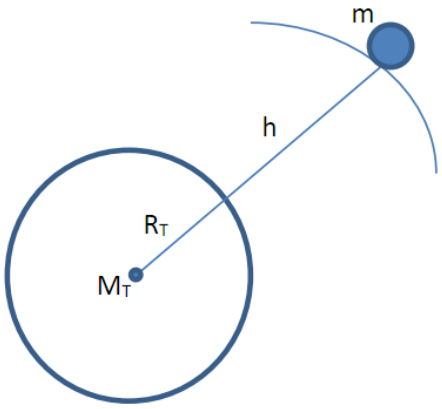
m: is the mass of the object

R_T : the radius of the Earth, equal to 6.37.10^6m.

g : is the norm of the gravitational acceleration due to the star of mass M.

If the body is located at a height h relative to the surface of the star of radius R, then :

$$F = \frac{m.M}{(R_t + h)^2}$$



$$g = \frac{GM}{(R_T + h)^2} = \frac{GM}{{R_T}^2}\frac{R^2}{(R_T + h)^2} = g_0\frac{R^2}{(R_T + h)^2}$$

g_0: is the gravitational acceleration gravitational acceleration on the surface of the star. Near the earths, surface its value 9.8 m/s^2

III.5.1.5. Kepler's third law in the case of circular motion:

The universal law of gravitation was established by Newton, relying in particular on Kepler's laws, including the third law: "The square of the period of revolution of a planet in the solar system is proportional to the cube of the semi-major axis of its orbit".

$$T^2 = kr^3$$

- **k** is a proportionality constant expressed in s^2/m^3. It is the same for all planets orbiting the Sun.

- **r** is the mean radius of the planet's orbit. In the case of an elliptical orbit, it represents the semi-major axis of the trajectory; for large planets, the trajectory is nearly circular and r corresponds to the radius of the circle.

Indeed, the orbital velocity, necessarily uniform in the case of such a circular motion, is:

$$v = \omega r = \frac{2\pi r}{T}$$

Moreover, knowing that we have equality of the centrifugal force with the centripetal gravitational force:

$$m\frac{v^2}{r} = G\ \frac{mM}{r^2}$$

Around a spherical body of mass M, we have the following relationship between the orbital period T and the radius of the circular orbit r of a satellite of very small mass m in front of M (G is the constant of universal gravitation) :

$$m\frac{\left(\frac{2\pi r}{T}\right)^2}{r} = G\ \frac{mM}{r^2}$$

$$\frac{4\pi^2 r^2}{T^2} = G\frac{M}{r}$$

Where:

$$T^2 = \frac{4\pi^2}{GM}r^3$$

By identifying with Kepler's third law, we obtain the following expression for the constant k:

$$K = \frac{4\pi^2}{GM}$$

Therefore, we have:

$$\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$$

III.5.1.6.The Geostationary Satellite:

A geostationary satellite is a satellite that an observer on Earth always sees in the same spot in the sky (it appears motionless relative to the ground). This characteristic is important for telecommunications and broadcasting. The satellite orbits the Earth at the same angular velocity as the Earth's rotation on its axis. As a result, its orbital period around the planet is the same as the planet's rotation period. Consequently, the geostationary satellite follows a uniform circular motion in a terrestrial. Frame with a period of T = 24 hours, and it lies within the Earth's equatorial plane. Recall that the period is as related to the angular velocity by:

Newton's second law gives, in magnitudes:

$$\begin{cases} F_g = G\dfrac{M_T m}{r^2} \\ a = \dfrac{V^2}{r} = \omega^2 r \end{cases} \Rightarrow G\frac{M_T m}{r^2} = m\omega^2 r$$

Where m_T is the mass of the Earth (5, 98·10^{24} kg), and m is the mass of the satellite. So, r:

$$\boldsymbol{r = \left(G\frac{M_T}{\omega^2}\right)^{1/3}}$$

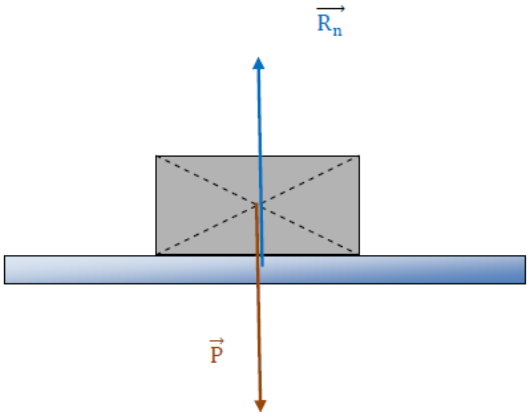
III.6. Contact forces:

The contact force is any force that takes place once we have contact between two flat surfaces.

III.6.1. Reaction of a support:

Consider an object placed on a solid support (horizontal surface). The object cannot penetrate into the support: there is a force called the reaction of the support and denoted R or N, which opposes it. This reaction is as applied to the object at the level of the object –support contact, and its direction is orthogonal to the surface of the support at the level of the contact.

The object being in equilibrium: $\vec{R} + \vec{P} = \vec{0} \Rightarrow \vec{R} = -\vec{P}$

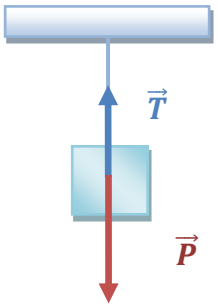


N: represents the resultant of all actions exerted on the contact surface.

III.6.2. Tension force:

A tension force is a force developed in a rope or spring when it is stretchy by an applied force. Tension is exerts along the entire length of the rope/spring in a direction opposite to the force applied to it.

Tension can also sometimes called stress, strain, or strain.



III.7. Frictional force:

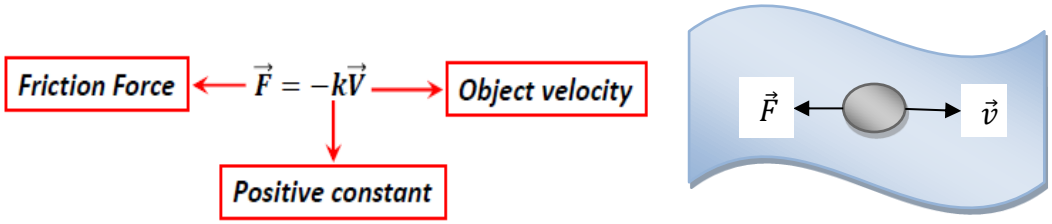
The frictional force results from the movement of a solid body on a solid surface (solid friction) or from the movement of a solid body on a liquid surface (viscous friction),

Whenever there is contact between two rough surfaces of two solid bodies, resistance arises and opposes the relative motion of the two bodies. There are sevra types of friction:

- Friction between solid bodies, which can be static or dynamic,
- Friction in fluides.

III.7.1. Viscous rubbing:

In this type of friction, the force is proportional to the speed (Figure)



We give

K**:** Depends on the geometric shape of the body

η**:** Fluid viscosity coefficient, depends on internal fluid friction,

III.7.2. Friction force:

Solid friction refers to the action of a solid object (the support) in contact with the system being Studied. These frictional forces exist only if:

- The system is moving relative to the support that exert the frictional forces (friction or Kinetic friction). In this case, the resultant of the solid frictional forces opposes the system's movement.
- The system is stationary relative to the support that exerts the frictional forces but is Subject to external forces that tend to move it (friction or static friction). In this case, the resultant of the solid frictional forces opposes the resultant of these external forces. It is

therefore opposite to the movement the system would have if there were no friction. In the case where solid friction forces exist, the reaction $\vec{F}$ of the support can be decomposes into

- A normal reaction to the support $\vec{R}$ which prevents the solid from sinking and which Always exists as long as there is contact.

III.7.2.1. Static friction force:

Static friction is the force that keeps a body at rest even in the presence of an external force. Where we have:

$$f_s = \mu_S R$$

The body is at rest: $\sum \vec{F} = \vec{0}$

By projecting onto the two axes Ox and Oy:

$\vec{F}.\cos\theta - f_s = 0$

$\vec{F}.\sin\theta + R - P = 0$

So, we can write: $\vec{f}.\cos\theta \ and \ R = P - F\,.\sin\theta$

- The coefficient of friction is as called static when the body is stationary.
- The coefficient of static friction is a ratio between the static frictional force of an object and the normal force, and is written as follows: $\mu_s = tan\,\alpha_0$

$$\mu_s = \frac{f_s}{R}$$

μ_s: Coefficient of static friction.
R: the reaction force, which represents the normal component of the contact force.

The static maximal is calculates by:

$$f_{smax} = \mu_s R = \mu_s (P - F\,sin\,\theta)$$

III.7.2.2. Kinetic friction force:

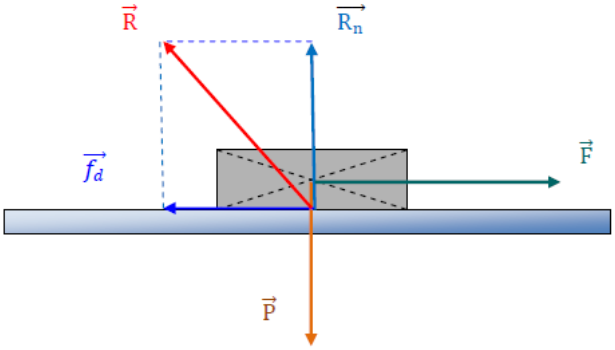
Kinetic or dynamic friction is the frictional force present when an object is in motion on another object.

Here the kinetic friction force becomes $f_d = \mu_d R$

By applying the fundamental principle of dynamics (Newton's second law), we have:

$$\sum \vec{F} = m.\vec{a}$$

$$\vec{F} + \vec{R} + \vec{P} + \overrightarrow{f_d} = m.\vec{a}$$



The projection on the axes (ox) and (oy) gives:

$$F.\cos\theta - f_d = m.a$$

$$F.\sin\theta + R - P = 0$$

So, we can write:

$$f_d = F.\cos\theta - m.a \; and \; R = P - F.\sin\theta$$

- The dynamic friction coefficient is a ratio between the dynamic friction force of an object and the normal force.

The coefficient of dynamic friction written as:

$$\boldsymbol{\mu_d = \frac{f_d}{R}}$$

Note:

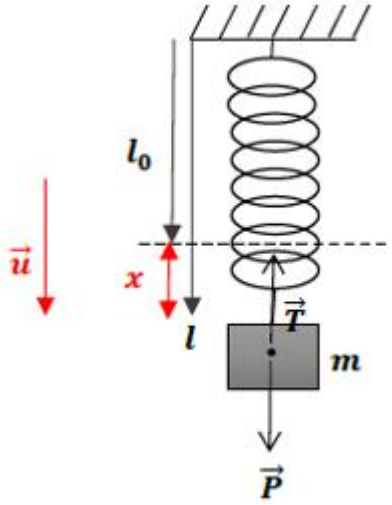
Experience has shown that the coefficient of static friction is greater than the coefficient of dynamic friction $\boldsymbol{\mu_s > \mu_d}$.

III.7.2.3. Spring Restoration Force:

Consider a point M of mass m attached to the end of a straight elastic pendulum (spring). When the spring stretches, a restoring force $\vec{\boldsymbol{F}} = -\boldsymbol{k}(\boldsymbol{l} - \boldsymbol{l_0})\vec{\boldsymbol{\mu}}$, proportional to this stretch and called tension, is exerts on the mass:

The notations commonly used and important to know are as follows:

- Unloaded or unloaded spring: spring neither compressed nor stretched (no deformation).
- l_0: Length of the spring at rest or unloaded.
- l: length of the deformed or stretched spring.
- K: Spring constant.
- Δl: Algebraic elongation of the spring, i.e., the difference between its current length l (deformed spring) and its upstretched length l_0.
- $\vec{\mu}$: Unit vector directed along the spring and oriented from the fixed end of the spring towards the free end capable of movement.



Where *l* denotes the length of the spring at the time considered at equilibrium, we have:

$$\sum \vec{F}_{ext} = \vec{0}$$
$$m\vec{g} - k(l - l_0)\vec{u} = \vec{0}$$
$$mg = k(l - l_0)$$

In motion, we apply the fundamental principle of dynamics: we choose the origin of the vertical axis Ox at the equilibrium position at time t, and the displacement of the mass relative to its equilibrium position is $l(t) = l_0 + x$. The projection of the fundamental principle of dynamics along this axis leads to:

$$mg - k(l(t) - l_0) = ma$$
$$mg - k(l - l_0 + x) = ma$$
$$-k(x) = ma \equiv m\ddot{x} + kx = 0$$
$$m\ddot{x} + kx = 0$$

We set $k/m = \omega^2$, the differential equation of motion of the mass m is:

$$\ddot{x} + \omega^2 x = 0$$

The solution of this homogeneous second-order differential equation is:

$$x(t) = A\cos(\omega t + \varphi)$$

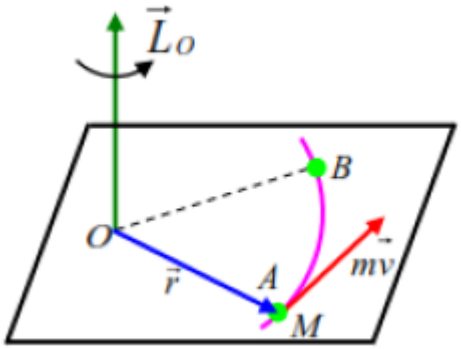
A, φ are two constants to be determined by the initial conditions.

III.8. Angular Momentum of a material point ($\vec{L}$) :

Consider a point mass M rotating about a fixed point in the Galilean frame of reference R (O, x, y, and z). Its velocity in this frame of reference is denoted $\vec{v}$. The angular momentum of point M with respect to a fixed point O is defines as the moment of its momentum with respect to O, that is:

$$\overrightarrow{L_0} = \vec{r} \wedge \vec{P}$$
$$\Rightarrow \overrightarrow{L_{/0}} = \overrightarrow{OM} \wedge m\vec{V}$$

The angular momentum $\vec{L}$, the cross product of the position vector $\overrightarrow{OM}$ with the momentum vector $\vec{P} \wedge m\overrightarrow{V}$, is a vector perpendicular to OM and to the velocity of the point: it is a quantity perpendicular to the trajectory of the point M see figure.



The unit of angular momentum is $Kg.m^{-2}.S^{-1}$

Special Cases:

The angular momentum is a vector perpendicular to the plane containing the vectors $\vec{r}$ and $\vec{P}$.

- If the movement is circular with radius r, we will have

$\vec{r} \perp \vec{v}$ and$v = \omega r$

$\vec{L}_o = \vec{r} \wedge m\vec{v} \Longrightarrow \vec{L}_o = r.mv.sin\frac{\pi}{2} = rmv = mr^2\omega$

$\vec{L}_o = mr^2\vec{\omega}$

- For a curvilinear plane movement, we use polar coordinates, with pole O:

$\overrightarrow{v} = \dot{r}\overrightarrow{u_r} + r\dot{\theta}\overrightarrow{u_\theta}$

$\vec{L}_o = \vec{r} \wedge m\vec{v} = m\vec{r} \wedge \vec{v} = m\vec{r} \wedge (\dot{r}\,\overrightarrow{u_r} + r\dot{\theta}\,\overrightarrow{u_\theta}) = mr2^2\dot{\theta}\vec{k} \Longrightarrow \vec{L}_o = mr2^2\theta$

III.8.1. Angular Momentum Theorem:

In many cases, it is more convenient to use the angular momentum theorem than Newton's Second Law. The time derivative of the angular momentum of a point mass M with respect to a fixed point O in a Galilean frame of reference (R) is equal to the sum of the moments with respect to the same point O of the external forces applied to the point M.

$$\frac{d\vec{L}_{/0}}{dt} = \frac{d(\overrightarrow{OM} \wedge m\vec{V})}{dt}$$

$$\frac{d\vec{L}_{/0}}{dt} = \left(\frac{d\overrightarrow{OM}}{dt} \wedge m\vec{V}\right) + \left(\overrightarrow{OM} \wedge \frac{d(m\vec{V})}{dt}\right)$$

$$\Rightarrow \frac{d\vec{L}_{/0}}{dt} = \left(\vec{V} \wedge m\vec{V}\right) + \left(\overrightarrow{OM} \wedge m\frac{d(\vec{V})}{dt}\right) = \overrightarrow{OM} \wedge m\frac{d\vec{V}}{dt}$$

$\vec{V}$ It is collinear with $m\overrightarrow{V}$, which gives $\vec{V} \wedge m\overrightarrow{V} = 0$

The previous equation becomes

$$\boldsymbol{\frac{d\vec{L}_{/0}}{dt} = \overrightarrow{OM} \wedge m\vec{a} = \overrightarrow{OM} \wedge \sum \vec{F} = \mathcal{M}_{/0}(\vec{F})}$$

➢ For the study of dynamical systems, these two theorems and the fundamental principle of dynamics, denoted PFD, are generally used.

➢ For a point mass rotating about a fixed axis, one can apply either the fundamental principle of dynamics or the angular momentum theorem.

➢ In Cartesian coordinates, we obtain the following expressions:

$$\vec{L}_{/O} = m\left(yV_z - zV_y\right)\vec{\imath} + m\left(zV_x - xV_z\right)\vec{\jmath} + m\left(xV_y - yV_x\right)\vec{k}$$

$$\overrightarrow{\mathcal{M}}_{/O}(\vec{F}) = m\left(yF_z - zF_y\right)\vec{\imath} + m\left(zF_x - xF_z\right)\vec{\jmath} + m\left(xF_y - yF_x\right)\vec{k}$$

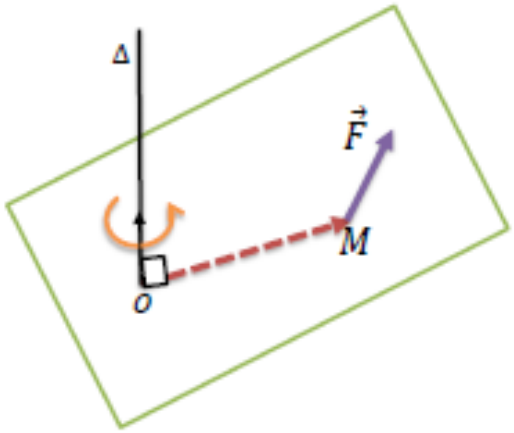
III.8.2. Moment of a force at a point:

Let be a particle M of mass m located at a point identified by the position vector $\vec{r} = \overrightarrow{OM}$ and subjected to a force $\vec{F}$. The moment of $\vec{F}$ relative to the point O is defined by :

$\overrightarrow{OM}_O\overrightarrow{(F)} = \overrightarrow{OM} \times \vec{F}$

$$\overrightarrow{\mathcal{M}}_{/O}(\vec{F}) = \overrightarrow{OM} \wedge \vec{F}$$

$$\overrightarrow{\mathcal{M}}_{/O}(\vec{F}) = \vec{0} \Rightarrow \vec{L}_{/O} = C^{ste}.$$



- If the force $\vec{F}$ is **parallel** to $\overrightarrow{OM}$ then it has no effect on the rotation of M.
- If the force $\vec{F}$ is **perpendicular** to $\overrightarrow{OM}$ then the moment of the force is maxi

The unit of the moment of a force is the N.m

III. 8.3. Application:

➢ **Simple pendulum:**

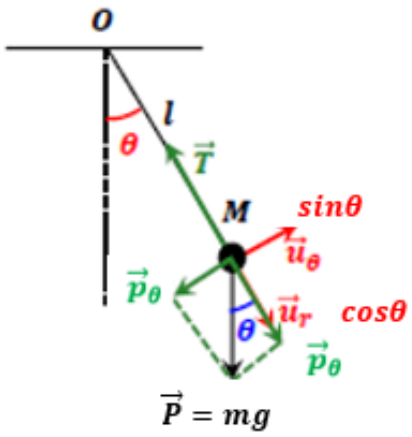
We consider a material point of mass m connected to a fixed point O of a Galilean reference frame by means of a wire of negligible mass and of length L.

The motion of this simple pendulum is assumes to occur in a vertical plane. The mass m is subject to two forces.

The motion of this simple pendulum is assumes to be in a vertical plane. The mass m is subjects to two forces: its weight $\vec{P}$ and the tension in the string$\vec{T}$.

Since the motion of M is planar, we have two unknowns, ρ (t) and θ (t), which are the polar coordinates of point M with point O as the origin and the downward-pointing vertical axis as the polar axis. In our case, it is clear that with the string taut, ρ (t) must equal L.

Therefore, only one unknown, θ(t), remains, which can be obtained using either the fundamental principle of dynamics, the conservation of total mechanical energy (because, fortunately, the tension in the string does no work), or the conservation of angular momentum.



The forces applied to the mass m are the weight $\vec{P}$ and the tension$\vec{T}$, their expressions in the polar coordinates is given:

$$\begin{cases} \overrightarrow{OM} = l\vec{u}_r \\ \vec{T} = -T\vec{u}_r \\ \vec{P} = P\vec{u}_r + P\vec{u}_\theta \end{cases}$$

$$\vec{P} = P\,cos\theta\,\vec{u}_r - P\,sin\,\theta\,\vec{u}_\theta \Rightarrow \overbrace{mg\,cos\theta\,\vec{u}_r}^{P\vec{u}_r} - \overbrace{mg\,sin\,\theta\,\vec{u}_\theta}^{P\vec{u}_\theta}$$

The equation of motion for a simple pendulum $\boldsymbol{\theta}\,(\boldsymbol{t})$ using the theorem of angular momentum

$$\frac{d\vec{L}/o}{dt} = \sum \vec{M}/o(\vec{F}_{ext})$$

And

$$\vec{M}/o(\vec{F}_{ext}) = \vec{M}/o(\vec{P}) + \vec{M}/o(\vec{T}) = \overrightarrow{OM} \wedge \vec{P} + \overrightarrow{OM} \wedge \vec{T}$$

With:

$$\vec{L}/o = \overrightarrow{OM} \wedge \vec{P}$$

$$\vec{L}/o = \overrightarrow{OM} \wedge m\vec{v} = (l\vec{u}_r) \wedge m(l\dot{\theta}\vec{u}_\theta)$$

$$\Rightarrow \vec{L}/o = ml^2\dot{\theta}(\vec{u}_r \wedge \vec{u}_\theta)$$

Such as: $\begin{cases} \vec{u}_r \wedge \vec{u}_r = 0 \\ \vec{u}_r \wedge \vec{u}_\theta = \vec{k} \end{cases}$

$$\Rightarrow \vec{L}/o = ml^2\dot{\theta}\vec{k}$$

On the other hand, we have:

$$\begin{cases} \vec{M}/o(\vec{F}_{ext}) = \overrightarrow{OM} \wedge \vec{P} = (l\vec{u}_r) \wedge (P\cos\theta\,\vec{u}_r - P\sin\theta\,\vec{u}_\theta) \\ \vec{M}/o(\vec{F}_{ext}) = \overrightarrow{OM} \wedge \vec{T} = (l\vec{u}_r) \wedge (-T\vec{u}_r) = \vec{0} \end{cases}$$

$$\vec{M}/o(\vec{F}_{ext}) = \overrightarrow{OM} \wedge \vec{P} + \overrightarrow{OM} \wedge \vec{T} = (l\vec{u}_r) \wedge mg(\cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta) + (l\vec{u}_r) \wedge (-T\vec{u}_r)$$

$$\Rightarrow \vec{M}/o(\vec{F}_{ext}) = -mgl\sin\theta\,\vec{k}$$

So the equation: $\frac{d\vec{L}/o}{dt} = \sum \vec{M}/o(\vec{F}_{ext})$

$$\Rightarrow \frac{d}{dt}\left(ml^2\dot{\theta}\vec{k}\right) = -mgl\sin\theta\,\vec{k}$$

$$\Rightarrow ml^2\ddot{\theta} = -mglsin\theta \Rightarrow l\ddot{\theta} + gsin\theta = 0$$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}sin\theta = 0$$

For small oscillations, we have: $\sin\theta \approx \theta$

$$\Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0$$

Proposed exercises about chapter III

Exercise1:

Calculate the gravitational field of a body of mass m:

1) On the surface of the earth,

2) At a height h from the earth.

m_T = *5.98×1024kg, the* mass of the earth

R_T = *6.37×106 m* the radius of the earth.

Exercise 2:

Consider a small block of mass m abandoned without initial velocity at point A of an inclined plane at an angle α=30° to the horizontal. Point A is at height h.

1- What is the value of the coefficient of static friction μs that keeps the mass in equilibrium at point A.?

Exercise 3:

A block of mass m ascends along a plane inclined by an angle α, with respect to the horizontal, with initial velocity v_0, and coefficient of friction μ_c.

1. Determine how far the block travels before coming to rest.

2. What is the maximum value that the static friction coefficient μ_s can take for the body to remain stationary.

3. For a value of the dynamic friction coefficient $\mu_c$lower than the maximum value found in the second question, what is the velocity v_1 of the body when it returns to its starting position?

Exercise 4:

A piece of ice M of mass m slides frictionless over the outer surface of an igloo, which is a half-sphere of radius r with a horizontal base.

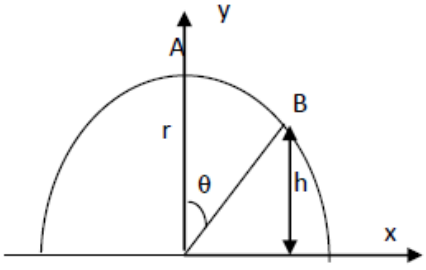
At t = 0, it is released from point A without any initial velocity.

1- Find the expression for the velocity at point B, as a function of g, r and θ.

2- Using the fundamental relation of dynamics, determine the expression of $|\vec{N}|$ the reaction of the igloo on M at point B as a function of velocity v_B.

3- At what height does M leave the sphere?

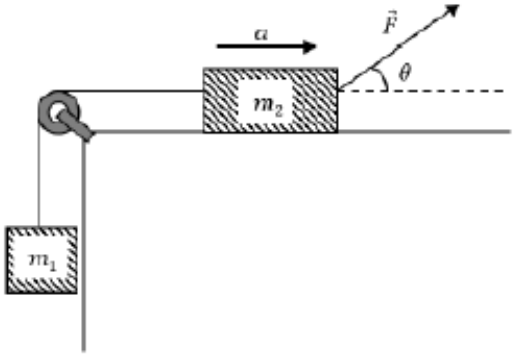
4- At what speed does M arrive at the axis (Ox)?



Exercise 5:

A block of mass m_1 assimilated to a material point can slide on a horizontal surface with a coefficient of kinetic friction μ_d one of its ends is as connected by an inextensible wire of negligible mass passing through a pulley of negligible mass connected to a second mass m_2.

A force of modulus F is as applied to m_2 at an angle θ to the horizontal. Find the accelerations of the two masses.



Exercise 6:

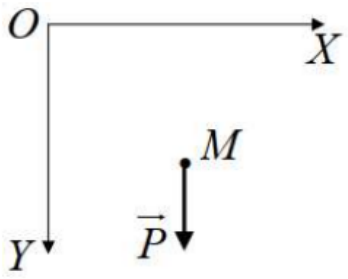
A material point M with mass m vibrates around a horizontal axis OZ perpendicular to the vertical plane (OX, OY) of the depicted movement. It position is defined at each instant by its Cartesian coordinates.

Calculate:

1) The moment of the weight vector $\vec{P}$ with respect to point O, and then with respect to the axis OZ as a function of x, g, and m.

2) The angular momentum of point M with respect to point O, and then with respect to the axis OZ as a function of m, x, y, $\dot{x}$ and $\dot{y}$

3) Find the equation of motion by applying the theorem of angular momentum to point M.



Correction of exercises about chapter III

Exercise 1:

1- On the surface of the earth:

$$F_g = G\frac{m_t m}{r_t^2} = mg \Rightarrow g = G\frac{m_t}{r_t^2}$$

$$g = 6.67 \times 10^{-11}\frac{5.98\times10^{24}}{(6.37\times10^{6})^2} = 9.82\ m.s^{-2}$$

$$g = 9.82\ m.s^{-2}$$

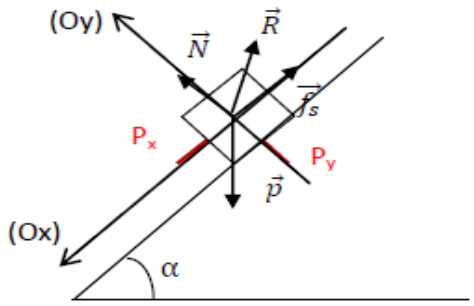
2- At a height h from the earth.

$$F_{g'} = G\frac{m_t m}{(r_t + h)^2} = mg \Rightarrow g' = G\frac{m_t}{(r_t + h)^2}$$

$$F_{g'} = G\frac{m_t}{(r_t + h)^2}$$

Exercise 2:

What is the value of the coefficient of static friction μs that keeps the mass in equilibrium at point A?



$$\sum \overrightarrow{F_{ext}} = \vec{0} \Rightarrow \vec{N} + \vec{f_s} + \vec{P} = \vec{0} \Rightarrow \vec{R} + \vec{P} = \vec{0}$$

Following (Ox): $-f_s + p_x = 0 \Rightarrow f_s = m\ g\ \sin\alpha$

(Oy): $N - p_y = 0 \Rightarrow N = m\ g\ \cos\alpha$

The maximum value that the coefficient of static friction μ_s can take is tgα.

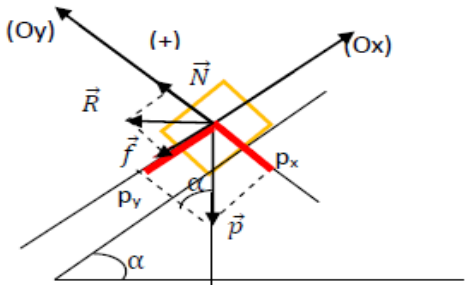
We have

$$tg\varphi = \frac{f_s}{N} = \mu_s = \frac{m\ g\ \sin\alpha}{m\ g\ \cos\alpha} = tg\alpha$$

Exercise 3:

According to the fundamental principle of dynamics:

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{R} = \vec{p} + \vec{N} + \vec{f} = m\vec{a}$$



1. Distance traveled before stopping

Acceleration along the plane:

$a = - g (\sin(\alpha) + \mu_c \cos(\alpha))$

Using kinematics:

$v^2 = v_0^2 + 2 a x$

$0 = v_0^2 - 2 g (\sin(\alpha) + \mu_c \cos(\alpha)) x$

Final expression:

$x = v_0^2 / [2 g (\sin(\alpha) + \mu_c \cos(\alpha))]$

2. Condition for equilibrium (static friction)

$mg \sin(\alpha) \leq \mu_s m g \cos(\alpha)$

$\mu_s \geq \tan(\alpha)$

3. Velocity when returning to the starting point

Energy relation during descent:

$v_1^2 = 2 g x (\sin(\alpha) - \mu_c \cos(\alpha))$

Substitute x:

$x = v_0^2 / [2 g (\sin(\alpha) + \mu_c \cos(\alpha))]$

$v_1^2 = v_0^2 \times (\sin(\alpha) - \mu_c \cos(\alpha)) / (\sin(\alpha) + \mu_c \cos(\alpha))$

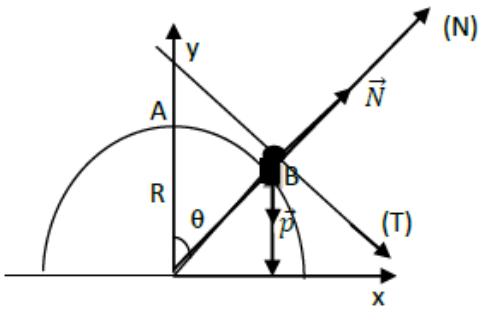
Final expression:

$v_1 = v_0 \sqrt{[(\sin(\alpha) - \mu_c \cos(\alpha)) / (\sin(\alpha) + \mu_c \cos(\alpha))]}$

Exercise 4:

The point at which the point leaves the sphere

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{p} + \vec{N} + \vec{f} = m\vec{a}$$



Followig (N): $N - p_N = -m.a_N \Rightarrow N - m\,g\cos\theta = -m\,v^2/R$ (**)

Following (T): $p_T = m.a_T \Rightarrow m\,g\sin\theta = m\,(dv/dt)$ (*)

$v = R\omega \Rightarrow \frac{dv}{dt} = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2}$ with $\omega = \frac{d\theta}{dt} = \dot{\theta}$ and $a_T = R\ddot{\theta} = R\frac{d\dot{\theta}}{dt}$

Equation (*) is multiplied by $\dot{\theta}$:

$$\Rightarrow \dot{\theta}\,g\sin\theta = \dot{\theta}R\frac{d\dot{\theta}}{dt} \quad \text{so} \quad \frac{d\theta}{dt}g\sin\theta = \dot{\theta}R\frac{d\dot{\theta}}{dt}$$

then $d\theta\,g\sin\theta = \dot{\theta}Rd\dot{\theta} \Rightarrow g\int_0^{\theta}\sin\theta\,d\theta = R\int_0^{\dot{\theta}}\dot{\theta}d\dot{\theta}$

$$g(1-\cos\theta) = R\frac{\dot{\theta}^2}{2} \Rightarrow 2g(1-\cos\theta) = \frac{v^2}{R} \quad \text{because } v^2 = R^2\dot{\theta}^2$$

$$\Rightarrow v^2 = 2(gR - gR\cos\theta)$$

The speed at point B is:

$$v_B = \sqrt{2gR(1-\cos\theta)}$$

1- The expression for the reaction of the igloo on M

$$N = m\,g\cos\theta - m\,a_N = m\,(g\cos\theta - 2g(1-\cos\theta))$$

$$\Rightarrow N = m\,(3\,g\cos\theta - 2g)$$

-The material point leaves the half-sphere at point p, so N=0

$$N=0 \Rightarrow mg\,(3\,\cos\theta_0 - 2) = 0 \text{ so } \cos\theta_0 = \frac{2}{3}$$

The height h at which the material point leaves the half-sphere is:

$$h_p = R\cos\theta = \frac{2}{3}R$$

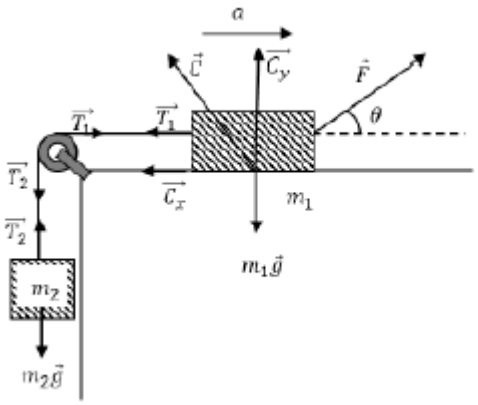
1- The velocity of the material point at this point:

$$2g(1-\cos\theta_0)=\frac{v_p^2}{R}\Rightarrow v_p^2=2gR(1-\cos\theta_0)$$

$$v_p=\sqrt{2gR(1-\cos\theta_0)}=\sqrt{\frac{2}{3}Rg}$$

Exercise 5:

The fundamental principle of dynamics is applied to the masses m_1 and m_2 :



For system m_1 :

$$\Sigma\vec{F}=m_1\vec{a}\Rightarrow\overrightarrow{p_1}+\overrightarrow{C_1}+\overrightarrow{T_1}+\vec{F}=m_1\vec{a}\ (1)$$

For system m_2 :

$$\Sigma\vec{F}=m_2\vec{a}\Rightarrow\overrightarrow{p_2}+\overrightarrow{T_2}=m_2\vec{a}\ (2)$$

The pulley with negligible mass, so $T_1=T_2=T$

Projecting equations (1) and (2) onto the direction of motion gives:

$$-C_x+F\cos\theta-T_1=m_1.a \quad (3)$$

and

$$T_2-m_2\,g=m_2.a \quad (4)$$

Projecting equation (1) onto the direction perpendicular to the motion gives:

$C_y=m_1\,g$

And $$\mu_d=C_x/C_y=F_f/R_N\Rightarrow C_x=\mu_d\,m_1\,g \quad (5)$$

Summing equations (3) and (4), assuming $T_1=T_2$ and equation (5), we obtain:

$$a=\frac{F\cos\theta-g(m_2-\mu_d\,m_1)}{m_1+m_2}$$

Exercise 6:

1) The moment of the weight vector $\vec{P}$ with respect to point O:

$$\left.\begin{matrix}\overrightarrow{\tau_0} = \overrightarrow{(OM} \wedge \vec{P}) \\ \vec{p} = \vec{p}_x + \vec{p}_y = mg.\vec{\jmath}\end{matrix}\right] \Rightarrow \overrightarrow{\tau_0} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ x & y & 0 \\ 0 & mg & 0 \end{vmatrix}$$

$$\overrightarrow{\tau_\Delta} = mgx\vec{k}$$

- The moment of the weight vector $\vec{P}$ with respect to the axis OZ = Δ :

$$\tau_\Delta = \left(\overrightarrow{OM} \wedge \vec{P}\right).\vec{k}$$

$$\tau_\Delta = mgx$$

2) The angular momentum of point M with respect to point O:

$$\left.\begin{matrix}\overrightarrow{L_0} = \overrightarrow{OM} \wedge \vec{P} \\ \vec{p} = m\vec{v}_x + m\vec{v}_y\end{matrix}\right] \Rightarrow \overrightarrow{L_0} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ x & y & 0 \\ \dot{x} & \dot{y} & 0 \end{vmatrix}$$

$$\overrightarrow{L_0} = m(x\dot{y} - y\dot{x}).\vec{k}$$

-The angular momentum of point M with respect to the axis OZ:

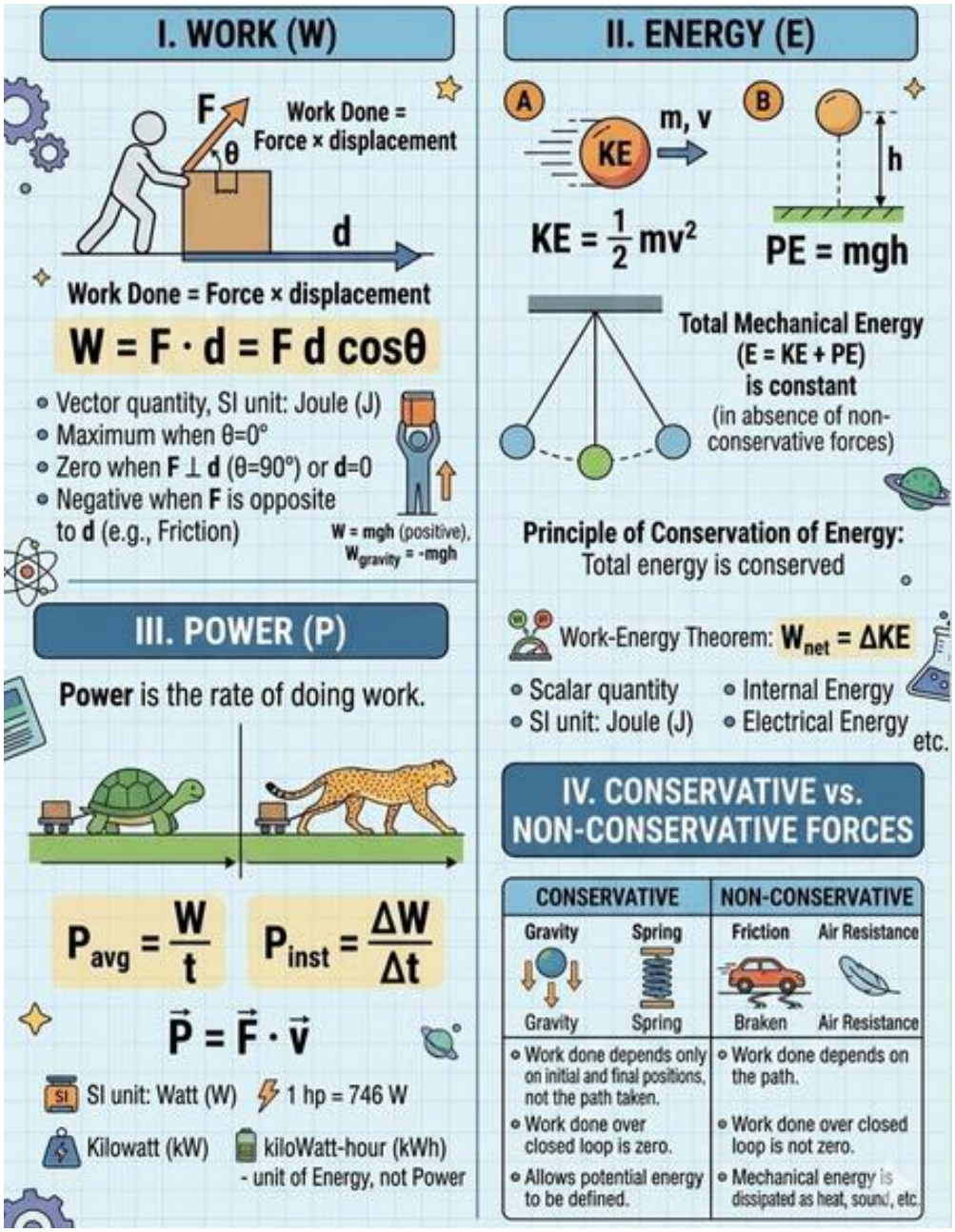
$$L_\Delta = \left(\overrightarrow{OM} \wedge \vec{P}\right).\vec{k}$$
$$\overrightarrow{L_0} = m(x\dot{y} - y\dot{x})$$

3) The equation of motion by applying the theorem of angular momentum to point M:

$$\frac{d\vec{L}_O}{dt} = \vec{\tau}_O$$

$$m(\dot{x}\dot{y} + x\ddot{y} - \dot{y}\dot{x} - y\ddot{x}).\vec{k} = mgx.\vec{k} \Rightarrow x\ddot{y} - y\ddot{x} = gx$$

CHAPTER IV: WORK, POWER AND ENERGY



Chapter IV: Work, Power and Energy

IV.1. Introduction:

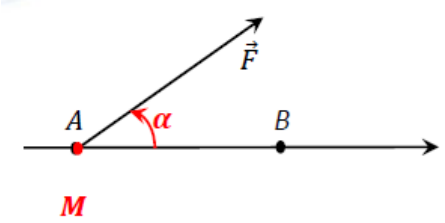
The concept of energy is one of the most important topics in science and engineering. In everyday life, we think of energy in terms of fuel for transportation and heating, electricity for lights and appliances.

We first introduce the concept of work. A force acting on an object does work when the point of application of that force moves through some distance. All motion under the action of external forces $\vec{F}$ implies work by these forces. In other words, work supplied by a force moves a body in its own direction and creates motion.

IV.1.1.Work performed by a constant force:

The work of a constant force $\vec{F}$ when it moves from point A to point B is as given by the scalar product.

$$W_{AB} = \vec{F} \times \overrightarrow{AB}$$



Where W_{AB} is work, its unit of measurement is the joule.

$\overrightarrow{AB}$ Is the displacement vector from point to point

Using the definition of a scalar product, we can write

$$w_{\vec{f}} = \vec{F}.\overrightarrow{AB} = F\,.AB\cos\alpha$$

φ is the angle between the two vectors $\vec{F}$ and $\overrightarrow{AB}$.

Note:

1- The work of the force is not relates the path followed, that is, the work W_{AB} of the force $\vec{F}$ has the same value regardless of the path.

2- If $\overrightarrow{AB}$ ⍰ $\vec{F}$ *then* $W_{AB}(\vec{F}) = 0$

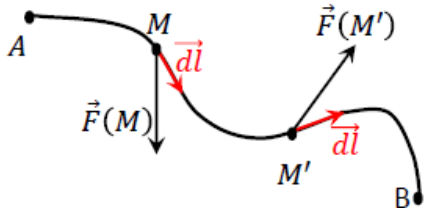
3- If $\boldsymbol{W_{AB} > 0}$, we say that work is a motive, and this is when $0 < \alpha < 90^{\circ}$.

4- If $\boldsymbol{W_{AB} < 0}$, we say that the work is resistant, and this is when $90 < \alpha < 180^{\circ}$.

Unity of work in the system MKSA is « Joule ».

IV.1.2.The work performed by a variable force:

If the force varies in intensity and/or direction during displacement, and if the displacement has any form whatsoever, we need to use integral calculus to generalize the definition of work. The work of a force depends on the path followed, which is why this elementary work is necessary.



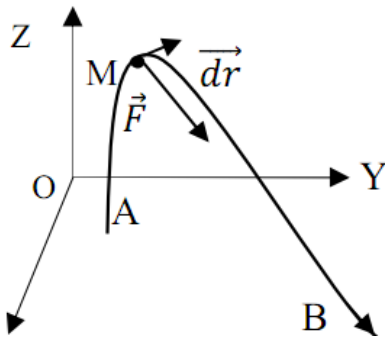
$$dW = \vec{F} \cdot d\vec{r} = \vec{F} \cdot d\vec{l}$$

The total work during the elemental transition from point to point is the sum of the elemental works . In addition, we write

$$W_{AB} = \int_A^B dW = \int_{r_A}^{r_B} \vec{F} \cdot d\vec{r}$$

We use this relationship if the force and elemental displacement is defined follows

$$\begin{cases} \vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} \\ d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k} \end{cases}$$

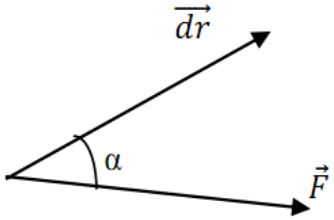


So,

$$dW = \vec{F} \cdot d\vec{r} = F_x dx + F_y dy + F_z dz$$

The elementary work dW performed by a force$\vec{F}$ on a point mass m during an elementary displacement ***dr= dl*** is given by

$$dW = \vec{F} \cdot d\vec{r} = |\vec{F}|.|d\vec{r}| \cos(\vec{F}, d\vec{r})$$



IV.2.The power:

Let a point M move along its trajectory at a velocity $\vec{v}$ (M) relative to the reference frame of study, it experiences a force $\vec{F}$ (M) as shown in the figure opposite:

The power of a force $\vec{F}$ is the work per unit time.

We have two types:

We know the instantaneous power by the following relationship:

$$\boldsymbol{p = \frac{dw}{dt}}$$

We know the average power by the following relationship:

$$\boldsymbol{p_{avr} = \frac{\Delta w}{\Delta t}}$$

Then the instantaneous power of the $\vec{F}$ is

$$\mathbf{P(\vec{F}) = \frac{dW}{dt} = \frac{|\vec{F}| \cdot |d\vec{r}|}{dt} = \vec{F} \cdot \vec{v}(M) = \|\vec{F}\| \times \|\vec{v}(M)\| \times \cos \alpha}$$

IV.3. Energy:

Energy, in physics, is the capacity for doing work. Energy is not a material substance.

Energy can neither be created nor destroyed it can only be transformed from one form to another energy

IV.3.1. Kinetic energy:

We apply the basic principle of motion with the expression elemental work.

We have $\boldsymbol{dW = \vec{F}d\vec{r}}$ and $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt}$

From this expression, we can deduce the following:

$dW = \vec{F}d\vec{r} = m\frac{d\vec{v}}{dt}d\vec{r}$ $\quad d\vec{r} = \vec{v}dt$

$$dW = m\frac{d\vec{r}}{dt}d\vec{v} = m\vec{v}d\vec{v}$$

$$W = m\int_A^B \vec{v}d\vec{v}$$

In the end we find

$$W_{AB}(\vec{F}) = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

The $\frac{1}{2}mv^2$ is the energy associated with the motion of the particle called the kinetic energy of (KE) thus, if a particle of constant mass m is moving with a speed v, its KE is a scalar quantity defined as:

$$\boldsymbol{K_E = \frac{1}{2}mv^2}$$

The unit of energy is the « Joule ».

In addition, since $p = mv$, we can write:

$$\boldsymbol{K_E = \frac{p^2}{2m}}$$

IV.3.1.1. Theorem of the Kinetic Energy:

The variation in kinetic energy of a material point subjected to a set of external forces between two positions A and B is equal to the sum of the work of these forces between these two points.

$$\boldsymbol{\Delta E_c = E_c(B) - E_c(A) = \sum W_{A-B}(\vec{F}_{ext})}$$

The theory of kinetic energy states that the change in the kinetic energy of a body between positions and is equal to the work of the net force applied to the body. In addition, we write

$$\Delta E_c = \sum W(\overrightarrow{F})$$

IV.3.1.2. Conservative force and non-conservative force:

A force is said to be conservative if its work between two points A and B does not depend on the path followed (W_1, W_2, and W_3), but only from the starting point A and the ending point B.

Any conservative force derives from a potential function $E_p\ (x, y, z)$ such that

$$\vec{F} = -\overrightarrow{grad} E_p(x, y, z)$$

Examples of conservative forces:

Gravitational forces, elastic forces, gravitational forces...

- Force is said to be neoconservative when their work depends on the path taken

***Example of non-conservative forces*:** Frictional forces.

IV.3.2. Potential energy:

By definition, the work done by conservative forces does not depend on the path taken but only on the initial and final states. The work done by these forces can expressed using a state function called potential energy E_p

Potential energy is the potential function associated with the conservative force.

- ✓ Potential energy is the energy related to position.
- ✓ Potential energy is defined up to a constant; it is always referred to a reference frame

Taken as the origin to calculate it.

- ✓ The work of a conservative force is related to the potential energy by the expression:

$$W_A^B = \int_A^B \vec{F} \cdot d\vec{r} = E_{PA} - E_{PB} = -\Delta E_P$$

$$\Delta E_P = E_{PA} - E_{PB}$$

$$\Delta W = -\Delta E_P \Rightarrow dW = -dE_P$$

E_P a toujours une référence prenant l'exemple du sol $E_{P0} = 0$

$E_P = mgh$, si le corps se trouve à une hauteur h du sol.

IV.3.2.1. Calculating the elastic potential energy:

From the previous example when we calculated the work of the spring's return force. We have got

$\vec{F} = -kx\vec{i}$

$\overrightarrow{dl} = dx \cdot \vec{i}$

$dW = \vec{F} \cdot \overrightarrow{dl}$

$dW = -dE_p = -kx.dx \Rightarrow dE_p = kxdx$

$$E_p(x) = -\int_0^x -kxdx = \frac{1}{2}kx^2$$

When F passes from a position x_1 to x_2, we have:

$$W_{\vec{F}} = \int_{x_1}^{x_2} \vec{\mathrm{F}} . \overrightarrow{dl} = \int_{x_1}^{x_2} -Kx \, \mathrm{d}x$$

$$= -\frac{1}{2}K(x_2^2 - x_1^2)$$

In the general case, we write the expression for the elastic potential energy of the spring

$$E_p(\Delta x) = \frac{1}{2}k\Delta x^2$$

IV.3.2.2. Calculating the potential energy of a body under the influence of the force of Gravity:

IV.3.2.2.1. Potential Energy of the Force of Gravity:

Returning to the free fall example.

Since the force of gravity on the axis is non-existent.

$$W_{A\to B}(\vec{\mathrm{F}}_{\mathrm{C}}) = -\Delta E_p$$

And $W = mg(y_1 - y_2) = mgh$

So: $W_{\vec{p}} = -\Delta E_p = -(E_{pf} - E_{pi}) = Epi = mg(z_A - z_B)$

Because E_{pf} is the reference potentiale nergy .

So: $Ep = mgH$

E_P to each other's previous reference to the same sample $E_{P0} = 0$

$E_P = mgh$, if the body comes into contact with the sun

IV.3.2.2.2. Potential Energy of a Gravitational Force:

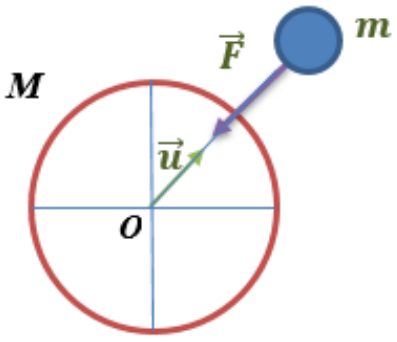
Consider an object with mass m moving under the influence of the gravitational force $\vec{F}_r$ exerted by the Earth on it, which is as given by:

$$\vec{F}(r) = -\frac{GMm}{r^2}\vec{u} = -\frac{GMm}{r^3}\vec{r} \quad \left(\vec{u} = \frac{\vec{r}}{r}\right)$$

$$\vec{F}(r) = -\overrightarrow{grad}E_P(r) = -\frac{dE_P(r)}{dr}\vec{u}$$

$$\Rightarrow \frac{dE_P(r)}{dr} = \frac{GMm}{r^2} \Rightarrow dE_P(r) = \frac{GMm}{r^2}dr$$

$$E_P(r) = \int \frac{GMm}{r^2}dr = -\frac{GMm}{r} + Cte$$



IV.3.3. Total energy or mechanical energy:

The mechanical energy of a material point at a given instant is equal to the sum of kinetic energy and potential energy:

$$E_T = E_K + E_p = E_m$$

Two examples:

- The potential energy of a system composed of a spring with spring constant k, whose extension is equal to at time t, under the action of a particle of mass m and instantaneous velocity v, is:

$$E_M = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

- In the case of freefall:

$$E_M = \frac{1}{2}mv^2 + mgz$$

IV.3.3.1. Mechanical Energy Theorem:

Based on the results of the relation, we deduce the following theorem: The change in mechanical energy of a system between two points A and B is equal to the sum of the work done by the non-conservative forces applied to the system between these two points.

$$\Delta E = E(B) - E(A) = \sum W_{A\to B}\left(\vec{F}_{ex}^{NC}\right)$$

Note:

Since non-conservative forces are resistive forces (their work being negative; W <0), the mechanical energy of a system can only decrease over time.

IV.3.3.2. Conservative System and Conservation of Mechanical Energy:

A system is said to be conservative if it is only subjected to conservative external forces Since the system is not subjected to any non-conservative forces, frictional forces are zero or negligible.

The system is also say to be "mechanically isolated".

IV.3.3.3. Mechanical Energy Theorem for a Conservative System:

The mechanical energy of a conservative (or mechanically isolated) system is conserved over time Conservative system ⇒ $E_M = K_E + P_E = E_0$ = constant.

$\Delta E = E(B) - E(A) = \sum W_{A\to B}\left(\vec{F}_{ex}^{NC}\right) = 0 \Rightarrow \frac{dE}{dt} = 0$ **⇒This constitutes the principle of conservation of mechanical energy.**

$\Delta E = 0 \Rightarrow E = C^{ste}$.

Proposed exercises about chapter IV

Exercise 1:

Let a material point M be subject to a force field F.

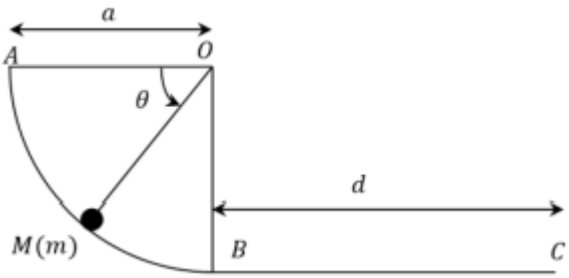
$\vec{F} = (x - ay)\,\vec{\imath} + (3y - 2x)\,\vec{\jmath}$

1. Calculate the work of the force F for the displacement of M from the point O (0, 0) to the point A (2, 4) passing through the point C (0, 4).

2. Determine the work of F for the displacement of M following a trajectory circular with radius R and center O (0, 0).

Exercise 2:

A particle of mass m, initially at rest in A, slides without friction on the circular surface AOB of radius a.



1) Determine the work of weight from A to M.
2) Determine the work of the surface-particle contact force m.

3) Determine the potential energy E_p of m at the point *M* (E_p (*B*) = 0).

4) Use the kinetic energy theorem to determine the speed of m at point M, deduce its kinetic energy E_c.
5) Calculate the mechanical energy E_m.

Exercise 3:

A solid body S of mass m is linked on one side to a spring of stiffness K, while the other side of the spring is fixed. The body is moves horizontally from its equilibrium position by a distance x and then released (μ =tgφ: coefficient of friction).

1. Show the forces applied to body S.

2. Calculate the speed V_B corresponding to the movement of S from its equilibrium position.

Exercises correction of about chapter IV

Exercise1:

The work done W by a force field $\overrightarrow{F} = P(x,y)\vec{I} + Q(x,y)\vec{j}$ along a path C is given by the line integral:

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (P\,dx + Q\,dy)$$

We know that:

$dW = \vec{F} \cdot d\vec{l}$

$d\vec{l}$ Is the elementary displacement

$\overrightarrow{dl} = dx\vec{i} + dy\vec{j}$

$\vec{F} = F_x\vec{i} + F_y\vec{j} = (x - ay)\vec{i} + (3y - 2x)\vec{j}$

$dW = F_x.dx + F_y.dy$

$dW = (x - ay).dx + (3y - 2x).dy$

1. **Work along the path O (0, 0) → C (0, 4)→A (2, 4)**

$W_{OCA} = W_{OC} + W_{CA}$

Segment 1: From O (0, 0) to C (0, 4)

• Path Characteristics: Vertical line along the y-axis, where x = 0 and dx = 0.

• Limits of Integration: y changes from 0 to 4.

• Calculation:

The path from O→C: $\begin{cases} x = 0 \Rightarrow dx = 0 \\ y\ varies\ from\ 0 \rightarrow 4 \end{cases}$ Q

$$W_{OC} = \int_0^4 dW = \int_0^4 (3y - 2(0)).dy = \int_0^4 3y.dy = \frac{3y^2}{2}\Big|_0^4 = \frac{3(4)^2}{2}$$

$$W_{OC} = 24$$

Segment 2: From C (0, 4) to A (2, 4)

• Path Characteristics: Horizontal line where y = 4 (constant), so dy = 0.

• Limits of Integration: x changes from 0 to 2.

• Calculation:

The path from AC→A: $\begin{cases} x\ varies\ from\ 0 \rightarrow 2 \\ y = 4 \Rightarrow dy = 0 \end{cases}$

$$W_{CA} = \int_0^2 dW = \int_0^2 (x - a(4)).dx = \int_0^2 x.dx - \int_0^2 4adx = \frac{x^2}{2}\Big|_0^2 - 4ax|_0^2$$

$$W_{CA} = \frac{2^2}{2} - 4a(2) = 2 - 8a$$

$$W_{CA} = 2 - 8a$$

Total Work for Question 1:

$$W_{OCA} = W_{OC} + W_{CA} = 24 + 2 - 8a = 26 - 8a$$

$$W_{OCA} = 26 - 8a$$

2. The work of F for the displacement of M following a trajectory circular with radius R and center 0(0,0):

$$\vec{F} = (x - 2y)\vec{i} + (3y - 2x)\vec{j}$$

$\vec{F}$ Is conservative force $\Rightarrow \vec{F}$ derives from a potential E_p

$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k} = -\overrightarrow{grad}\, E_p = -\frac{\partial E_p}{\partial x}\vec{i} - \frac{\partial E_p}{\partial y}\vec{j} - \frac{\partial E_p}{\partial z}\vec{k}$$

$$(x - 2y)\vec{i} + (3y - 2x)\vec{j} = -\frac{\partial E_p}{\partial x}\vec{i} - \frac{\partial E_p}{\partial y}\vec{j}$$

$$\begin{cases} \frac{\partial E_p}{\partial x} = -(x - 2y) \dots \dots (1) \\ \frac{\partial E_p}{\partial y} = -(3y - 2x) \dots (2) \end{cases}$$

$$E_p = -\frac{x^2}{2} + 2xy - \frac{3y^2}{2} + C$$

$$from\ (1) \Rightarrow E_p = -\int (x - 2y)dx = -\frac{x^2}{2} + 2xy + C(y) \dots \dots (3)$$

$$from\ (3) \Rightarrow \frac{\partial E_p}{\partial y} = 2x + \frac{dC(y)}{dx} \dots \dots \dots (4)$$

$$(4) = (2) \Rightarrow 2x + \frac{dC(y)}{dx} = -(3y - 2x) \Rightarrow \frac{dC(y)}{dx} = -3y$$

$$C(y) = -\frac{3y^2}{2} + C$$

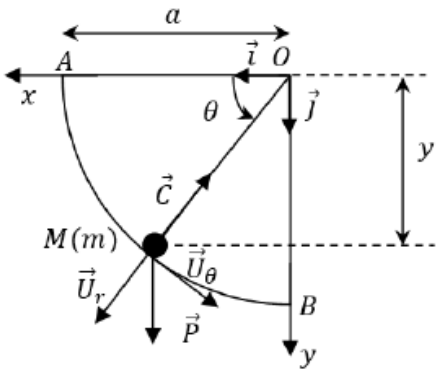
Exercise 2:

1) The work of $\vec{p}$ from A to M is:

$dW = \vec{p}.\overrightarrow{dl}$ with $p = mg\vec{j}$

$\overrightarrow{dl} = dx\vec{i} + dy\vec{j}$ so $dW = mgdy$

$$W = mg\int_0^y dy = mgy = mg\ a\sin\theta$$



2) The work of R_N force is:

$W_R = \int_0^y \overrightarrow{R_N}.\overrightarrow{dl} = \vec{0}$ Because $\overrightarrow{R_N} \perp \overrightarrow{dl}$

3) Potential energy:

$dEp = -dW \Rightarrow Ep = -mg\ a.\sin\theta + c$

$E_p(B) = 0,\ \theta = \pi/2$

$\Rightarrow E_p = mga(1-\sin\theta)$

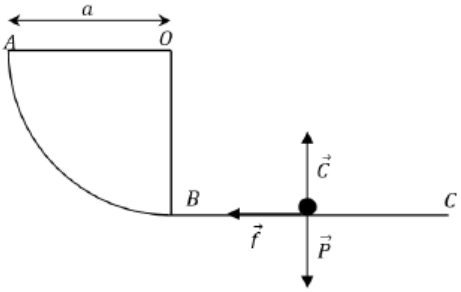
4) $\Delta E_C = \sum W \Rightarrow \frac{1}{2}mv_M^2 = mga\sin\theta$

$$V_M = \sqrt{2ga\sin\theta}$$

5) $E_m = E_c + E_P = mg\ a = cste$

6) E_P decreases E_c increases while E_m remains constant.

7) The coefficient of kinetic friction:



$\mu = \frac{f}{R_N} = \frac{f}{p} \Rightarrow f = \mu mg$

So $\Delta E_C = W_f = \int_B^C \vec{f}.\overrightarrow{dl} = -\mu mgd \Rightarrow \frac{1}{2}mv_B^2 = -\mu mgd$

$$v_B = \sqrt{2ag}$$

We cane used also:

$E_{m_B} = E_{m_A} \Rightarrow E_{C_A} + E_{P_A} = E_{C_B} + E_{P_B}$

Calculation of μ:

$$\sum \vec{F} = m\vec{\gamma} = \vec{f} + \vec{P} + \overrightarrow{R_N}$$

We have:

$\mu = \frac{f}{R_N} = \frac{f}{mg}$ because $R_N = mg$ (with projection on (oy))

Projection on (ox): $-f = m.\gamma$

$v_C^2 - v_B^2 = 2\gamma.d$ ($v_C = 0$)

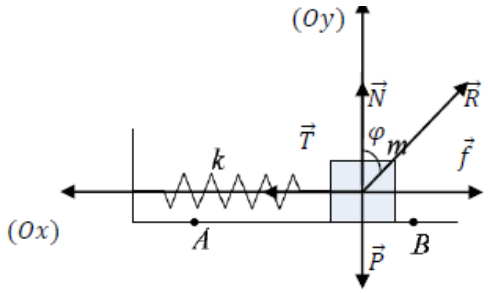
$-v_B^2 = 2\gamma.d = -2ag$ so: $\gamma = \frac{-ag}{d}$ with $-f = m.\gamma = \frac{-mag}{d}$ so $f = \frac{m.a.g}{d}$

Then

$\mu = \frac{f}{R} = \frac{f}{mg} = \frac{mag}{mg.d} = \frac{a}{d} = \frac{1}{3}$

Exercise 3:

1. Show the forces applied to body S.



2- Calculate the speed V_B corresponding to the movement of S from its equilibrium position.

$\Delta Ec = \Sigma W_{f_{ext}} \Rightarrow Ec_B - Ec_A = W_p + W_T + W_N + W_f$

With, $W_{\vec{p}} = W_{\vec{R}} = \vec{0}$ because $\vec{R}$ *and* $\vec{p} \perp \overrightarrow{Ox}$

So $Ec_B = W_T + W_f$

$(Ec_A = 0$ *because the initial velocity is null*$)$

We have,

$$\vec{F} = -kx\vec{i} \text{ and } \overrightarrow{dl} = dx\vec{i}$$

$$\Rightarrow W_{\vec{F}} = \int dW_{\vec{F}} = \int \vec{F}.\overrightarrow{dl} = -k\int_x^0 xdx = \frac{1}{2}kx^2$$

so $W_T = \frac{1}{2}kx^2$ and $W_f = -f\,x$

According to the fundamental principle of dynamics:

$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{N} + \vec{p} + \vec{T} + \vec{f} = m\vec{a}$

$$\mu = tg\varphi = \frac{f}{N}$$

We choose a frame of reference composed of the (Ox) axis following the axis of motion, i.e. in the same direction as $\vec{T}$ and the (Oy) axis following $\vec{N}$.

By projecting onto the Oy axis

$N = mg$ *and* $\mu = tg\varphi = \frac{f}{mg}$

So f=mg tg φ

$\Rightarrow \frac{1}{2}mv_B^2 = \frac{1}{2}kx^2 - \text{mg x } \mu$

Hence,

$$v_B^2 = \frac{2}{m}\left(\frac{1}{2}kx^2 - \text{mg x } \mu\right)$$

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